Locality of Gapped Ground States in Systems with Power-Law-Decaying Interactions

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It has been proved that in gapped ground states of locally interacting lattice quantum systems with a finite local Hilbert space, the effect of local perturbations decays exponentially with distance. However, in systems with power-law- $(1/r^{\alpha})$ decaying interactions, no analogous statement has been shown and there are serious mathematical obstacles to proving it with existing methods. In this paper, we prove that when α exceeds the spatial dimension D, the effect of local perturbations on local properties a distance r away is upper bounded by a power law $1/r^{\alpha_1}$ in gapped ground states, provided that the perturbations do not close the spectral gap. The power-law exponent α_1 is tight if $\alpha > 2D$ and interactions are two-body, where we have $\alpha_1 = \alpha$. The proof is enabled by a method that avoids the use of quasiadiabatic continuation and incorporates techniques of complex analysis. This method also improves bounds on ground-state correlation decay, even in short-range interacting systems. Our work generalizes the fundamental notion that local perturbations have local effects to power-law interacting systems, with broad implications for numerical simulations and experiments.

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I. INTRODUCTION AND OVERVIEW OF RESULTS

Locality is a fundamental principle that underlies many theories of nature. Loosely speaking, locality means that an object is influenced directly only by its immediate surroundings and, in particular, should be insensitive to actions taken far away. The precise quantitative statement of this principle takes different forms in different contexts. In quantum many-body dynamics, locality manifests itself in the form of a causality light cone: roughly, if a local perturbation takes place at time $t = 0$, then at time t its effect must be within a ball region $r \leq vt$, where r is the distance and v is the maximal allowed speed of propagation of any physical particles or signals in the system. In relativistic quantum field theories, such a causality light cone is guaranteed by Lorentz invariance, where v is the speed of light, and effects exactly vanish outside the light cone. In non-relativistic quantum many-body systems with short-range interactions, the Lieb-Robinson bound (LRB) [1] guarantees an effective causality light cone: the effect of local perturbations decays exponentially in $(r - vt)$, where the speed v depends on the microscopic details of the system [2–4].

The consequences of locality take a slightly different form for equilibrium properties of the quantum many-body system. An important case is on the effect of a local perturbation on ground states. Specifically, let $\hat{H}$ be the Hamiltonian and consider the effect of a local perturbation $\hat{V}_Y$ (supported on region Y) on a local observable $\hat{S}_X$, supported on a region X far from Y. Intuitively, we expect that the expectation value of $\langle \hat{S}_X \rangle$ measured in the perturbed ground state should not deviate significantly from its unperturbed value when the distance d_{XY} is large, i.e., the deviation

$$\delta\langle \hat{S}_X \rangle_{\hat{V}_Y} \equiv \langle \hat{S}_X \rangle_{\hat{H}+\hat{V}_Y} - \langle \hat{S}_X \rangle_{\hat{H}} \tag{1}$$

should be small in magnitude. This intuition is rigorously formulated as the principle that local perturbations perturb locally (LPPL) [5], which states that for gapped ground states of a locally interacting Hamiltonian, $|\delta\langle \hat{S}_X \rangle_{\hat{V}_Y}|$ is upper bounded by a subexponentially decaying function in d_{XY} [6], provided that the perturbation does not close

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the spectral gap. The proof is based on the idea of quasiadiabatic continuation (QAC) [7–9], which relates the perturbed ground state $|G\rangle_{\hat{H}+\hat{V}_Y}$ to the unperturbed one by a quasilocal unitary evolution

$$|G\rangle_{\hat{H}+\hat{V}_Y} = \mathcal{T} e^{i\int_0^1 \hat{H}_{\text{eff}}(t)dt}|G\rangle_{\hat{H}}, \tag{2}$$

where $\mathcal{T}$ is the time-ordering operation and the effective Hamiltonian $\hat{H}_{\text{eff}}(t)$ only contains interactions that are subexponentially localized near Y. This immediately transforms the problem back to the dynamical case, where a Lieb-Robinson bound implies that $|\delta\langle\hat{S}_X\rangle_{\hat{V}_Y}|$ decays subexponentially in d_{XY}. This bound has later been strengthened to an exponential decay $|\delta\langle\hat{S}_X\rangle_{\hat{V}_Y}| \leq Ce^{-\mu_1 d_{XY}}$ [10,11], where C is a constant and μ_1 is given in Table I.

In recent years, there has been increasing interest in understanding the analogous consequences of locality from long-range power-law- ($1/r^\alpha$) decaying interactions, driven in part by the ubiquity of these interactions in many cold-atom and molecule [12–16], Rydberg atom [17–23], and trapped-ion [24–28] experiments, typically with $0 \leq \alpha \leq 6$, as well as the Coulomb interaction. The important question then arises: when long-range interactions are present, to what extent can we still expect locality in the senses described above to hold? The answer to this question is far from obvious, since long-range interactions can give rise to nonlocal behaviors of correlation functions for sufficiently small α [29,30]. For the dynamical part, the LRB has been successfully generalized to power-law-interacting systems [31–37], implying generalized causality light cones ($r \propto e^{vt}$ for $D < \alpha < 2D$ [2], $r = vt^\beta$ for $2D < \alpha < 2D + 1$ [31], and $r = vt$ for $\alpha > 2D + 1$ [33,35]).

However, the implications of locality for equilibrium systems are far less understood when power-law interactions are present, even in the important case of gapped ground states. This is partly due to the difficulties caused by the appearance of long-range interactions in $\hat{H}_{\text{eff}}(t)$ in Eq. (2): QAC only leads to an LPPL bound for $\alpha > 2D$ [38], an extremely restrictive condition and one rarely satisfied in the experimental systems of interest. Furthermore, even for $\alpha > 2D$, the LPPL principle has never been proved and the above method with QAC in Ref. [38] would lead to power-law exponents in the resulting bounds that are not tight (for details, see Appendix A).

In this paper, we prove the LPPL principle for gapped ground states of lattice quantum systems where interactions are bounded by a power law $1/r^\alpha$ in distance r, with $\alpha > D$. To achieve this goal, we devise an alternative method that avoids the use of QAC [Eq. (2)] (thereby circumventing the aforementioned difficulty) and incorporates techniques of complex analysis. This method also improves the LPPL bounds for short-range interacting systems and applies to degenerate (either exact or approximate) ground states as well. Our main result is roughly as follows: for perturbations $\hat{V}_Y$ that do not close the spectral gap,

$$|\delta\langle\hat{S}_X\rangle_{\hat{V}_Y}| \leq \begin{cases} \mathbf{P}(\ln d_{XY})/d_{XY}^{\alpha_1}, \\ \mathbf{P}(d_{XY})e^{-\mu_1 d_{XY}}, \end{cases} \tag{3}$$

where $\langle\ldots\rangle$ is a uniform average over the (possibly degenerate) ground-state subspace; the first line is for power-law systems and the second line is for short-range interacting systems; the exponents α_1 and μ_1 are given in Table I; and throughout this paper, we use $\mathbf{P}(x)$ to denote a polynomial in x with non-negative coefficients [but $\mathbf{P}(x)$ in different equations or in different parts of the same equation need not be the same] [39]. We see that α_1 is equal to α if $\alpha > 2D$ and interactions are two-body, in which case our bound is qualitatively tight [up to the subleading prefactor $\mathbf{P}(\ln d_{XY})$] since it agrees with perturbation theory.

As one notable by-product, the method we use to obtain these bounds also improves bounds on correlation decay [2,40] of gapped (possibly degenerate) ground states: for arbitrary local operators $\hat{A}_X$ and $\hat{B}_Y$, their connected correlation function is bounded by

$$|\langle\hat{A}_X\hat{B}_Y\rangle - \langle\hat{A}_X\rangle\langle\hat{B}_Y\rangle| \leq \begin{cases} \mathbf{P}(\ln d_{XY})/d_{XY}^{\alpha_2} \\ \mathbf{P}(d_{XY})e^{-\mu_2 d_{XY}}, \end{cases} \tag{4}$$

where the exponents α_2 and μ_2 are given in Table I. We see that our method improves earlier exponents, even in the case of short-range interacting systems, where our bound improves that of Ref. [2] by approximately a factor of 2 for $\Delta \ll v$.

Our results have profound implications on numerical simulations and experiments. For example, it has been pointed out [41] that the LPPL principle straightforwardly implies an upper bound on the finite-size error (FSE) of several numerical ground-state algorithms, such as exact diagonalization [42,43] and the density matrix renormalization group [44,45], that is, the error in approximating an infinite system with a finite one. Our results [Eq. (3)] imply that the FSE of a local observable $\hat{S}$ in gapped ground-state simulations decays in the linear dimension of the system L as

$$\delta\langle\hat{S}\rangle_L \equiv |\langle\hat{S}\rangle_L - \langle\hat{S}\rangle_\infty| \leq \begin{cases} \mathbf{P}(\ln L)/L^{\alpha_3}, \\ \mathbf{P}(L)e^{-\mu_3 L}, \end{cases} \tag{5}$$

provided that the finite system is connected to the thermodynamic limit by a uniformly gapped path [41]. As in Eqs. (3) and (4), the first line is for power-law systems while the second line is for short-range interacting systems and the constants α_3 and μ_3 are given in Table I.

The paper is organized as follows. Table I summarizes the exponents α_1, α_2, α_3, μ_1, μ_2, and μ_3 in Eqs. (3), (4),

TABLE I. A summary of the constants α_1 and μ_1 (LPPL bounds), α_2 and μ_2 (correlation decay bounds), and α_3 and μ_3 (FSE bounds) for previous results compared with ours, for both power-law- and short-range-interacting systems. Our main result is the proof of the LPPL principle [Eq. (3)] for ground states of power-law-interacting systems with spectral gap Δ but we also significantly improve the bound for systems with exponentially decaying interactions, as well as the constants α_2 and μ_2 that appear in the correlation decay bounds [Eq. (4)]. The FSE bound [Eq. (5)] with exponents α_3 and μ_3 is a primary application of our main result [previously, there has been only a FSE bound for short-range systems [41], in which $\mu_3 = \mu_1 = \mu/(1 + 2\mu v/\Delta)$]. v is a constant that appears in the LRB and can be straightforwardly calculated (for short-range-interacting systems, v is the Lieb-Robinson speed).

	Prior bound		Our bound (LPPL and correlation decay have the same exponents: $\alpha_1 = \alpha_2,\ \mu_1 = \mu_2$)	
Interaction	LPPL	Correlation decay		FSE bound
$1/r^\alpha, \alpha > 2D$, two-body	–	$\alpha_2 = \alpha$ [36]	$\alpha_1 = \alpha$	$\alpha_3 = \alpha - D$
$1/r^\alpha, \alpha > D$	–	$\alpha_2 = \frac{\alpha}{1+2v/\Delta}$ [2]	$\alpha_1 = \frac{2\alpha}{\pi}\arcsin(\tanh\frac{\Delta\pi}{2v})$	If $\alpha > D+1$: $\alpha_3 = \min(\alpha - D, \alpha_1 + 1 - D)$ If $D < \alpha \le D+1$: $\alpha_3 = \begin{cases}\alpha - D, & \alpha_1 > D\\ \alpha_1 + \alpha - 2D, & \alpha_1 \le D\end{cases}$
$e^{-\mu r}$	$\mu_1 = \frac{\mu}{1+2\mu v/\Delta}$ [10]	$\mu_2 = \frac{\mu}{1+2\mu v/\Delta}$ [2,46]	$\mu_1 = \frac{2\mu}{\pi}\arcsin(\tanh\frac{\Delta\pi}{2\mu v})$	$\mu_3 = \mu_1$

and (5) for various interaction ranges. In Sec. II, we introduce our improved method: we use this method to bound the response of local observables in gapped nondegenerate ground states and to obtain the main result, Eq. (3). In Sec. III, we generalize the bounds to gapped degenerate ground states. In Sec. IV, we discuss the implications of our bounds in finite-size numerical simulations and prove Eq. (5). In Sec. V, we use our improved method to obtain tighter bounds on ground-state correlation decay, given in Eq. (4). We conclude in Sec. VI.

II. LOCALITY OF PERTURBATIONS TO GAPPED NONDEGENERATE GROUND STATES

Our setup is as follows. Let Λ_L be an infinite sequence of D-dimensional finite lattices, labeled by the linear system size $L \in \mathbb{Z}$, and $N \propto L^D$ is the number of lattice sites in total. On each site $i \in \Lambda_L$ sits a quantum degree of freedom with local Hilbert space $\mathcal{H}_i$. In this paper, we focus on fermionic systems or quantum spin systems where $\mathcal{H}_i$ is finite dimensional, although our formalism can be straightforwardly generalized to bosonic systems where $\dim(\mathcal{H}_i)$ is infinite. The Hamiltonian H_L acts on the global Hilbert space $\mathcal{H}_L \equiv \bigotimes_{i\in\Lambda_L}\mathcal{H}_i$ and can be written in the generic form

$$\hat{H}_L = \sum_{X\subset\Lambda_L} \hat{h}_X, \tag{6}$$

where the summation is over all subsets of Λ_L and $\hat{h}_X$ is the local Hamiltonian supported on X [47] (we later specify some locality condition on $\hat{h}_X$ that requires $\|\hat{h}_X\|$ to be small for large X). Throughout this section, we assume that $\hat{H}_L$ has a nondegenerate ground state $|G_L\rangle$ with spectral gap Δ_L (the energy difference between the first excited state and the ground state) that is uniformly bounded from below, i.e., there exists $\Delta^{(0)} > 0$ such that $\Delta_L \ge \Delta^{(0)}$ for all Λ_L. At this point, we do not make assumptions on the range of interaction; nor do we assume that the local Hilbert space is finite dimensional.

Let $\hat{V}_Y$ be a local perturbation supported on region Y. Suppose that for all $\lambda \in [0,1]$, $\hat{H}_L(\lambda) \equiv \hat{H}_L + \lambda\hat{V}_Y$ has a nondegenerate ground state $|G_L(\lambda)\rangle$ with spectral gap $\Delta_L(\lambda)$ that is uniformly bounded from below, i.e., $\exists\Delta > 0$ such that $\forall\lambda \in [0,1]$, $\Delta_L(\lambda) \ge \Delta > 0$, for all Λ_L. This condition will always be satisfied for sufficiently small perturbations satisfying $||\hat{V}_Y|| < \Delta^{(0)}/2$ ($||\cdots||$ is the operator norm), since Weyl's inequality [48] gives $\Delta_L(\lambda) \ge \Delta_L - 2\lambda||\hat{V}_Y|| \ge \Delta^{(0)} - 2||\hat{V}_Y||$.

Let $\hat{S}_X$ be a local observable supported on region X such that $X \cap Y = \emptyset$. Our goal is to bound the response of $\hat{S}_X$ to the local perturbation $\hat{V}_Y$, as defined in Eq. (1). We achieve this goal in two steps: in Sec. II A, we present a general method to bound $\delta\langle\hat{S}_X\rangle_{\hat{V}_Y}$ using a Lieb-Robinson-type bound on the unequal time correlator $\langle G_L(\lambda)|[\hat{S}_X(t), \hat{V}_Y]|G_L(\lambda)\rangle$, where $\hat{S}_X(t) = e^{i\hat{H}_L t}\hat{S}_X e^{-i\hat{H}_L t}$; and then, in Secs. II B–II D, we specialize to systems with different interaction ranges and apply the corresponding Lieb-Robinson bounds to obtain our main results in Eq. (3) and Table I. The resulting bounds are independent of the system size L, so they hold in the thermodynamic limit $L \to \infty$.

A. The improved method

In the following, we present an improved method to bound $\delta\langle\hat{S}_X\rangle_{\hat{V}_Y}$ using a Lieb-Robinson-type bound on

$\langle G_L(\lambda)|[\hat{S}_X(t),\hat{V}_Y]|G_L(\lambda)\rangle$. There are two main improvements compared to previous approaches: the first part generalizes the method in Ref. [41], which avoids QAC and directly relates $\delta\langle\hat{S}_X\rangle_{\hat{V}_Y}$ to a specially constructed correlation function, while the second part obtains a bound on this correlation function from a LRB on $|\langle G_L(\lambda)|[\hat{S}_X(t),\hat{V}_Y]|G_L(\lambda)\rangle|$ using complex-analysis techniques, which significantly improves the previous method in Ref. [41].

Since we have a gapped path for $\lambda\in[0,1]$, we can use perturbation theory to relate the rate of change of $\langle\hat{S}_X\rangle_{L,\lambda}\equiv\langle G_L(\lambda)|\hat{S}_X|G_L(\lambda)\rangle$ at each λ to a special correlation function, from which we obtain an exact expression for $\delta\langle\hat{S}_X\rangle_{\hat{V}_Y}$ as an integral over the correlation function. We choose the normalization and phase of $|G_L(\lambda)\rangle$ such that $\langle G_L(\lambda)|G_L(\lambda)\rangle=1$ and $\langle G_L(\lambda)|d/d\lambda|G_L(\lambda)\rangle=0,\forall\lambda\in[0,1]$. For any finite L, first-order nondegenerate perturbation theory gives the exact identity

$$\frac{d}{d\lambda}|G_L(\lambda)\rangle=\frac{\bar{P}_{G_L}(\lambda)}{\hat{H}_L(\lambda)-E_L(\lambda)}\hat{V}_Y|G_L(\lambda)\rangle, \tag{7}$$

where $E_L(\lambda)$ is the ground-state energy of $\hat{H}_L(\lambda)$ and $\bar{P}_{G_L}(\lambda)\equiv\hat{\mathbb{1}}-|G_L(\lambda)\rangle\langle G_L(\lambda)|$ is the projection operator to the subspace of excited states. Then,

$$\frac{d}{d\lambda}\langle\hat{S}_X\rangle_{L,\lambda}=\langle G_L(\lambda)|\hat{S}_X\frac{\bar{P}_{G_L}(\lambda)}{\hat{\Delta}_L(\lambda)}\hat{V}_Y|G_L(\lambda)\rangle+\text{c.c.}, \tag{8}$$

where $\hat{\Delta}_L(\lambda)\equiv\hat{H}_L(\lambda)-E_L(\lambda)$, the spectrum of which is lower bounded by Δ (in the subspace of excited states). In the following, we prove a uniform bound (independent of L and λ) on the rhs of Eq. (8), so that a bound on $\delta\langle\hat{S}_X\rangle_{\hat{V}_Y}$ immediately follows from $|\delta\langle\hat{S}_X\rangle_{\hat{V}_Y}|\le\int_0^1 d\lambda|d\langle\hat{S}_X\rangle_{L,\lambda}/d\lambda|$.

From now on, we omit the labels L,λ. We define

$$\Omega_{XY}(\omega)\equiv\langle G|\hat{S}_X\frac{i\bar{P}_G}{\omega-\hat{\Delta}}\hat{V}_Y|G\rangle-\langle G|\hat{V}_Y\frac{i\bar{P}_G}{\omega+\hat{\Delta}}\hat{S}_X|G\rangle \tag{9}$$

in the region $\omega\in\mathbb{C}\backslash K_\Delta$, where $K_\Delta=\{\omega\in\mathbb{R}|\omega\ge\Delta\text{ or }\omega\le-\Delta\}$. Note that for any finite system size L, $\Omega_{XY}(\omega)$ is a complex analytic function in its domain. Furthermore, the rhs of Eq. (8) is exactly $i\Omega_{XY}(0)$. For $|\text{Im}(\omega)|>0$, we have an integral representation for $\Omega_{XY}(\omega)$,

$$\Omega_{XY}(\omega)=\int_0^{\eta_\omega\infty}\langle G|[\hat{S}_X(t),\hat{V}_Y]|G\rangle e^{i\omega t}dt, \tag{10}$$

where $\eta_\omega=\text{sgn}[\text{Im}(\omega)]$. Taking the absolute value of Eq. (10) and using the triangle inequality, we have

$$\begin{aligned}|\Omega_{XY}(\omega)|&\le\int_0^\infty|\langle G|[\hat{S}_X(t),\hat{V}_Y]|G\rangle|e^{-|\text{Im}(\omega)|t}dt\\&\le\int_0^\infty C(d_{XY},t)e^{-|\text{Im}(\omega)|t}dt\\&\equiv\bar{\Omega}(d_{XY},y),\end{aligned} \tag{11}$$

where $y=|\text{Im}[\omega]|>0$. In the second line of Eq. (11), we assume a Lieb-Robinson-type bound $|\langle G|[\hat{S}_X(t),\hat{V}_Y]|G\rangle|\le C(d_{XY},t)$, the expression for which is given in Secs. II B–II D when we consider systems with different ranges of interaction. At large t, $C(d_{XY},t)$ equals the constant trivial bound $2\|\hat{S}_X\|\|\hat{V}_Y\|$, so $\bar{\Omega}(d_{XY},y)$ is finite for any ω with $\text{Im}(\omega)\neq0$ but diverges as $\bar{\Omega}(d_{XY},y)\sim1/y$ when $y\to0$, so gives no bound on the desired $|\Omega_{XY}(0)|$.

Nevertheless, we can obtain a bound on $|\Omega_{XY}(0)|$ from the above by using a powerful technique from complex analysis. The analyticity of $\Omega_{XY}(\omega)$ allows us to improve the bound on $|\Omega_{XY}(\omega)|$ over the initial bound in Eq. (11), by applying the following lemma [49, Theorem 2.12].

Lemma 1.—If $g(z)$ is complex analytic in a domain (a simply connected open region) S, then $u(z)=\ln|g(z)|$ is a subharmonic function in S, i.e., for any $z_0\in S$ and $\rho>0$, if the circular region defined by $|z-z_0|\le\rho$ is contained in S, then

$$u(z_0)\le\frac{1}{2\pi}\int_0^{2\pi}u(z_0+\rho e^{i\theta})d\theta. \tag{12}$$

Since $\Omega_{XY}(\omega)$ is complex analytic in the open disk region defined by $|\omega|<\Delta$, Lemma 1 implies that

$$\begin{aligned}\ln|\Omega_{XY}(0)|&\le\frac{1}{2\pi}\int_0^{2\pi}\ln|\Omega_{XY}[\rho e^{i\theta}]|d\theta\\&\le\frac{1}{2\pi}\int_0^{2\pi}\ln\bar{\Omega}[d_{XY},|\rho\sin\theta|]d\theta,\end{aligned} \tag{13}$$

for any $\rho\in(0,\Delta)$. We will see that the integration over θ in the last line is convergent despite $\bar{\Omega}(d_{XY},y)$ diverging when $y\to0$ [50].

The rest of our task is to insert the LRB of specific systems into Eq. (11) to obtain $\bar{\Omega}(d_{XY},y)$ and then compute the second line of Eq. (13) to obtain an upper bound for $|\Omega_{XY}(0)|$, which we do in Secs. II B–II D. In Appendix B 2, we introduce a technique to further improve the bound in Eq. (13) using a conformal mapping.

B. Power-law interactions with $\alpha>2D$

We start with the simplest case: $\alpha>2D$ and all interactions being two-body, i.e., all the $\hat{h}_X$ in Eq. (6) are of the form $\hat{h}_X=h_{ij}\hat{V}_i\hat{W}_j$ where $\hat{V}_i$ and $\hat{W}_j$ are local operators with unit norm and finite support separated by a distance d_{ij} and the h_{ij} are real parameters satisfying $h_{ij} \le \mathbf{C} d_{ij}^{-\alpha}$ [51]. Similar to the $\mathbf{P}(x)$ notation, throughout this paper we use $\mathbf{C}$ to denote a positive constant independent of r and t, and $\mathbf{C}$ in different equations, or in different parts of the same equation, need not be the same. In this case, we use the Hastings-Koma bound [2] for short times, the algebraic light-cone [52] Lieb-Robinson bound [31] for intermediate times, and the trivial bound for long times:

$$\|[\hat{S}_X(t), \hat{V}_Y]\| \le \begin{cases} \mathbf{C}e^{v't}/d_{XY}^{\alpha}, & 0 \le t \le t', \\ \mathbf{C}e^{vt - \mathbf{C}\frac{d_{XY}}{t^{\gamma}}} + \frac{\mathbf{C}t^{\alpha(1+\gamma)}}{d_{XY}^{\alpha}}, & t' < t \le t_0, \\ \mathbf{C}, & t > t_0, \end{cases} \tag{14}$$

where $t' = \alpha \ln \alpha / v$, $t_0 = \mathbf{C} d_{XY}^{1/(\gamma+1)}$, v is a constant, and $\gamma = (1+D)/(\alpha - 2D)$. Using $|\langle G|[\hat{S}_X(t), \hat{V}_Y]|G\rangle| \le \|[\hat{S}_X(t), \hat{V}_Y]\|$, and substituting $C(d_{XY}, t)$ in Eq. (11) by the rhs of Eq. (14) gives

$$\begin{aligned} \bar{\Omega}(r,y) &= \int_0^{t'} \mathbf{C}\frac{e^{(v'-y)t}}{r^{\alpha}} dt + \int_{t'}^{t_0} C(r,t) e^{-yt} dt \\ &\quad + \int_{t_0}^{\infty} \mathbf{C} e^{-yt} dt \\ &\le \frac{\mathbf{C}t_0}{2}[e^{-\mathbf{C}r} + e^{(v-y)t_0 - \mathbf{C}r/t_0^{\gamma}}] \\ &\quad + \frac{1}{r^{\alpha}}\left[\mathbf{C} + \mathbf{C}\frac{\Gamma[\alpha(\gamma+1)+1]}{y^{\alpha(\gamma+1)+1}}\right] + \mathbf{C}\frac{e^{-yt_0}}{y}, \end{aligned} \tag{15}$$

where for the second term in the rhs of the first line we use Jensen's inequality, since the integrand is convex when $t' \le t \le t_0$ (this convexity relies on a relation between the different constants here, which can always be satisfied; see Appendix B 1). The third line in Eq. (15) decays subexponentially in r, while the last line decays algebraically, so the term proportional to $r^{-\alpha}$ dominates the long-distance behavior of $\bar{\Omega}(r,y)$. Inserting $\bar{\Omega}(r,y)$ into Eq. (13), we obtain Eq. (3), where the subleading factor $\mathbf{P}(\ln r)$ is a constant in this case (for details, see Appendix B 1).

C. Power-law interactions with $\alpha > D$

The bound in the previous section does not apply to the case $D < \alpha < 2D$ and is limited to two-body (two-cluster) interactions. In this section, we consider the more general case where the $\hat{h}_X$ in Eq. (6) satisfies [2]

$$\sum_{X:X\supset\{i,j\}} \|\hat{h}_X\| \le \frac{h_0}{d_{ij}^{\alpha}}, \tag{16}$$

for all i and j, with $\alpha > D$. In this case, the Hastings-Koma bound [2] is the tightest general LRB:

$$C(r,t) \le \min\left\{\mathbf{C}\frac{e^{vt}-1}{r^{\alpha}}, \mathbf{C}\right\}, \tag{17}$$

where v is a positive constant. Inserting Eq. (17) into Eq. (11) gives

$$\begin{aligned} \bar{\Omega}(r,y) &= \int_0^{\infty} C(r,t)e^{-yt}dt \\ &\le \int_0^{t_0} \mathbf{C}\frac{e^{(v-y)t}}{r^{\alpha}}dt + \int_{t_0}^{\infty} \mathbf{C}e^{-yt}dt \\ &\le \mathbf{C}t_0 \frac{1+e^{(v-y)t_0}}{2r^{\alpha}} + \mathbf{C}\frac{e^{-yt_0}}{y} \\ &= \frac{\mathbf{C}t_0}{2r^{\alpha}} + \mathbf{C}\left(\frac{t_0}{2} + \frac{1}{y}\right)e^{-yt_0} \\ &\le \begin{cases} \mathbf{C}t_0 e^{-yt_0}/y, & y \le v, \\ \mathbf{C}t_0 r^{-\alpha}, & y > v. \end{cases} \end{aligned} \tag{18}$$

where we define $t_0 = (\ln \mathbf{C} + \alpha \ln r)/v$ and in the third line we use Jensen's inequality (due to the convexity of the integrand) to simplify the integral, rather than evaluating it exactly, in order to facilitate later computations.

We now insert Eq. (18) into Eq. (13), to upper bound $|\Omega_{XY}(0)|$. Equation (13) becomes

$$\begin{aligned} \ln|\Omega_{XY}(0)| &\le \ln \mathbf{C}t_0 - \frac{2}{\pi}\int_{\theta_0}^{\pi/2} \alpha \ln r d\theta \\ &\quad - \frac{2}{\pi}\int_0^{\theta_0}[t_0\rho\sin\theta + \ln(\rho\sin\theta)]d\theta, \end{aligned} \tag{19}$$

where $\theta_0 = \arcsin(v/\rho)$ if $v < \rho$ and $\theta_0 = \pi/2$ if $v \ge \rho$. Finishing this integral, and then taking the limit $\rho \to \Delta$, we obtain Eq. (3), with

$$\alpha_1 = \frac{2\alpha\Delta}{\pi v}(1 - \cos\theta_0) + \alpha\left(1 - \frac{2\theta_0}{\pi}\right). \tag{20}$$

In Appendix B 3, we improve this result using the technique of conformal mapping and obtain the result in Table I. We use the improved result [Eq. (B11)] for the rest of the paper.

D. Short-range interacting systems

The method in Sec. II A also significantly improves the LPPL bounds for systems with short-range interactions, either exponentially decaying or strictly finite ranged. Specifically, we consider systems the Hamiltonians of which [Eq. (6)] satisfy [2]

$$\sum_{X:X\supset\{i,j\}} \|\hat{h}_X\| \leq h_0 e^{-\mu d_{ij}}, \tag{21}$$

for all i and j, where μ is some positive constant. The Lieb-Robinson bound is [2]

$$C(r,t) \leq \mathbf{C}e^{-\mu(r-vt)}. \tag{22}$$

Note that the rhs of Eq. (22) can be obtained from the rhs of Eq. (17) with the substitutions $r \to e^r$, $\alpha \to \mu$, and $v \to \mu v$. We can therefore directly make this substitution in the results of Sec. II C and Appendix B 3 and obtain the bound

$$|\Omega_{XY}(0)| \leq \mathbf{P}(r)e^{-\mu_1 r}, \tag{23}$$

with μ_1 given in Table I. We see that for $\Delta \ll v$, our bound gives $\mu_1 \approx \Delta/v$, which improves the previous best bound $\mu_1 = \mu/(1+2\mu v/\Delta) \approx \Delta/(2v)$ by approximately a factor of 2. Furthermore, if we want a tighter bound for a specific model, we can use the LRB in Eq. (32) of Ref. [4]: $C(r,t) \leq \mathbf{C}e^{\omega_m(i\kappa)t-\kappa r}, \forall\kappa > 0$, where $\omega_m(i\kappa)$ is some (efficiently computable) function of κ (Ref. [4] mainly deals with systems with finite-range interactions but the method can be directly generalized to systems with exponentially decaying interactions). This leads to a bound of the same form as Eq. (23) in which μ_1 is a function of κ. We can then maximize $\mu_1(\kappa)$ over $\kappa > 0$. This method gives further quantitative improvement for a specific model, especially at large Δ/v.

III. GENERALIZATION TO GAPPED DEGENERATE GROUND STATES

In this section, we generalize our bounds to gapped systems with degenerate ground states. We begin with a straightforward extension. Note that if the system has a subspace $\mathcal{H}_1 \subseteq \mathcal{H}$ such that both the Hamiltonian H and the perturbation $\hat{V}_Y$ leave $\mathcal{H}_1$ invariant (this is not required for $\hat{S}_X$) and the ground state $|G_1\rangle$ of $\mathcal{H}_1$ is nondegenerate and gapped (within $\mathcal{H}_1$), then all our proofs in the previous section apply to this subspace $\mathcal{H}_1$, provided that $\bar{P}_G$ in Eq. (7) is understood as the projector to all the excited states within $\mathcal{H}_1$. In particular, if the system has a set of conserved quantum numbers that commute with both $\hat{H}$ and $\hat{V}_Y$ and distinguish all the gapped degenerate ground states, then our bounds apply to all the ground states.

Nevertheless, this simple extension does not apply if the perturbation $\hat{V}_Y$ breaks the conserved quantities. It also fails if the degeneracy is not due to any symmetry at all, which includes the important class of topological degeneracy, where the (approximately) degenerate ground states cannot be distinguished by local conserved quantum numbers. In the following, we present a more general treatment for degenerate ground states (motivated by the method in Ref. [53]), which shows that all our results in Table I still hold provided that $\langle\hat{S}_X\rangle$ is averaged over all the (nearly) degenerate ground states with equal weights. This can be thought of as the temperature $T \to 0$ limit of the statistical mechanical average, as long as this limit is taken after the thermodynamic limit $L \to \infty$, in which the splitting of ground-state degeneracy vanishes.

Let us denote the degenerate ground states of $\hat{H}(\lambda) = \hat{H} + \lambda\hat{V}_Y$ as $|G^a(\lambda)\rangle$, with energy $E_0^a(\lambda)$, for $a = 1, 2, \ldots, d$, respectively. Note that we do not require the degeneracy to be exact (which is important for treating topological degeneracy) but only that at each λ, all the ground-state energies $E_0^a(\lambda)$ are separated from the rest of the spectrum (the excited states) by at least an amount $\Delta(\lambda) > 0$ and $\Delta(\lambda)$ is uniformly bounded from below, i.e., $\Delta \equiv \inf_{\lambda\in[0,1]}\Delta(\lambda) > 0$. [Similar to the nondegenerate case, as long as $\Delta(0) > 0$, the uniform-gap condition is always satisfied for sufficiently small $\|\hat{V}_Y\|$, as guaranteed by Weyl's inequality.]

The method follows Sec. II A but now using degenerate perturbation theory. If some of the ground states are exactly degenerate at some λ, then we have some freedom to choose a basis for the exactly degenerate subspace and it can be shown that [53] it is always possible to choose a suitable basis for this subspace such that $\hat{V}_Y$ is diagonal within this subspace and $\langle G^a(\lambda)|\partial_\lambda|G^b(\lambda)\rangle = 0$ whenever $E_0^a(\lambda) = E_0^b(\lambda)$. Then, degenerate perturbation theory generalizes Eq. (7) to

$$\partial_\lambda|G^a(\lambda)\rangle = \frac{\bar{P}^a(\lambda)}{\hat{H}(\lambda) - E_0^a(\lambda)}\hat{V}_Y|G^a(\lambda)\rangle, \tag{24}$$

where

$$\begin{aligned}\bar{P}^a(\lambda) &= \mathbb{1} - \sum_{b:E_0^b(\lambda)=E_0^a(\lambda)} |G^b(\lambda)\rangle\langle G^b(\lambda)| \\ &= \bar{P}_G(\lambda) + \sum_{b:E_0^b(\lambda)\neq E_0^a(\lambda)} |G^b(\lambda)\rangle\langle G^b(\lambda)|,\end{aligned} \tag{25}$$

where $\bar{P}_G(\lambda) \equiv \hat{\mathbb{1}} - \sum_{b=1}^d |G^b(\lambda)\rangle\langle G^b(\lambda)|$ is the projection operator to the space of all excited states. Inserting the second line of Eq. (25) into Eq. (24), we obtain

$$\partial_\lambda|G^a(\lambda)\rangle = \frac{\bar{P}_G(\lambda)}{\hat{H}(\lambda) - E_0^a(\lambda)}\hat{V}_Y|G^a(\lambda)\rangle + \sum_{b=1}^d Q^{ab}|G^b(\lambda)\rangle, \tag{26}$$

where

$$Q^{ab} = \begin{cases} \frac{\langle G^b(\lambda)|\hat{V}_Y|G^a(\lambda)\rangle}{E_0^b(\lambda)-E_0^a(\lambda)}, & \text{if } E_0^b(\lambda) \neq E_0^a(\lambda), \\ 0, & \text{if } E_0^b(\lambda) = E_0^a(\lambda), \end{cases} \tag{27}$$

is an anti-Hermitian matrix $(Q^{ab})^* = -Q^{ba}$. We now consider the expectation value $\langle \hat{S}_X \rangle_\lambda$ of a local observable $\hat{S}_X$ averaged over all degenerate ground states $\{|G^b(\lambda)\rangle\}_{b=1}^d$, i.e., we define $\langle \hat{O} \rangle_\lambda \equiv 1/d \sum_{b=1}^d \langle G^b(\lambda)|\hat{O}|G^b(\lambda)\rangle$ for any operator $\hat{O}$. Then, Eq. (8) becomes

$$\partial_\lambda \langle \hat{S}_X \rangle_\lambda = \left\langle \hat{S}_X \frac{\bar{P}_G(\lambda)}{\hat{H}(\lambda) - E_0^a(\lambda)} \hat{V}_Y \right\rangle_\lambda + \text{c.c.}, \tag{28}$$

where, importantly, the contribution of the second term in Eq. (26) cancels due to anti-Hermiticity of Q^{ab}. The rest of Sec. II A generalizes in a straightforward way, with the only difference being that the ground-state expectation value $\langle G(\lambda)|\ldots|G(\lambda)\rangle$ is replaced by the average $\langle \ldots \rangle_\lambda$. Lieb-Robinson bounds can still be used as we have $\langle [\hat{S}_X(t), \hat{V}_Y] \rangle_\lambda \le \|[\hat{S}_X(t), \hat{V}_Y]\| \le C(r,t)$. All resulting bounds remain the same as those listed in Table I.

IV. IMPLICATIONS FOR FINITE-SIZE NUMERICAL SIMULATIONS

In this section, we present a straightforward application of our results, bounding the FSEs of local observables in gapped ground states of power-law systems and generalizing the bounds for locally interacting systems proved in Ref. [41]. The basic configuration for the one-dimensional (1D) case is illustrated in Fig. 1. The FSE for a local observable $\hat{S}_X$ measured in a L-site calculation is defined as $\delta \langle \hat{S}_X \rangle_L \equiv |\langle \hat{S}_X \rangle_L - \langle \hat{S}_X \rangle_\infty|$, which can be considered as the effect of the boundary interaction $\hat{V}_Y$ on $\hat{S}_X$, since removing $\hat{V}_Y$ from the thermodynamic Hamiltonian $\hat{H}$ decouples the finite system and the outside, leading to $\langle \hat{S}_X \rangle_L = \langle \hat{S}_X \rangle_{\hat{H} - \hat{V}_Y}$. We assume that the spectral gap $\Delta_L(\lambda)$ of the interpolated Hamiltonian $\hat{H} - \lambda \hat{V}_Y$ is uniformly bounded from below $\min_{\lambda \in [0,1]} \Delta_L(\lambda) = \Delta > 0$ [54]. Under this assumption, we can apply our main result, given in Eq. (3), to upper bound $\delta \langle \hat{S}_X \rangle_L$. A complication here is that $\hat{V}_Y$ contains infinitely many terms, including those that are very close to $\hat{S}_X$, so $r = d_{XY}$ is zero. To solve this issue, we can write

$$\hat{V}_Y = \sum_{i \in L, j \notin L} \hat{V}_{ij}, \tag{29}$$

where the summation is over all the interaction terms $\hat{V}_{ij}$ with i in the L-site system and j outside (here we overload the notation L to also denote the set of sites of the L-site system). Inserting Eq. (29) into Eq. (8) and using Eqs. (9)–(13) to upper bound the contribution of each individual $\hat{V}_{ij}$ term independently, we obtain

$$|\delta \langle \hat{S}_X \rangle_L| \le \sum_{i \in L, j \notin L} \|\hat{V}_{ij}\| \mathbf{P}(\ln r_{iX}) / r_{iX}^{\alpha_1}. \tag{30}$$

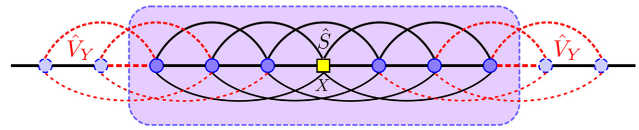

FIG. 1. Upper bounding the FSE with the LPPL, illustrated for a 1D chain. The LPPL principle immediately gives an upper bound on the FSE of local observables in numerical simulations of gapped ground states, by recognizing $\hat{V}_Y$ as the interactions between the sites of the finite system and sites lying outside.

In the following, we treat the 1D case for simplicity and present the derivation in arbitrary dimension in Appendix C 2. Letting $R = L/2$ and $\delta(r) = \mathbf{P}(\ln r)/r^{\alpha_1}$, we have

$$\begin{aligned} |\delta \langle \hat{S}_X \rangle_L| &\le \sum_{-R \le i \le R, |j| > R} \delta(|i|+1)/(j-i)^\alpha \\ &\le \sum_{-R \le i \le R} \mathbf{C} \delta(|i|+1)/(R+1-i)^{\alpha-1} \\ &\le \sum_{i=1}^{R+1} \mathbf{P}(\ln i) i^{-\alpha_1} (R+2-i)^{1-\alpha}. \end{aligned} \tag{31}$$

The following lemma gives a bound for the convolutional sum (for a proof, see Appendix C 1).

Lemma 2.—Let η and ζ be real constants satisfying $0 < \eta \le \zeta$. Then,

$$\sum_{r=1}^{R-1} \frac{\mathbf{P}(\ln r)}{r^\zeta (R-r)^\eta} \asymp \mathbf{P}(\ln R) \times \begin{cases} R^{-\eta}, & \text{if } \zeta \ge 1, \\ R^{1-\eta-\zeta}, & \text{if } \zeta < 1, \end{cases} \tag{32}$$

where the notation $f(R) \asymp g(R)$ means that there exist positive constants c_1, c_2 independent of R such that $c_1 g(R) \le f(R) \le c_2 g(R)$ for all $R \in \mathbb{Z}_{\ge 2}$.

Applying Lemma 2 to Eq. (31), we obtain Eq. (5) with

$$\alpha_3 = \begin{cases} \alpha_1 + \alpha - 2, & \text{if } \alpha_1 \le 1, \\ \alpha - 1, & \text{if } \alpha_1 > 1, \end{cases}$$

for $1 < \alpha \le 2$ and $\alpha_3 = \alpha - 1$ for $\alpha > 2$, which is the result in Table I for $D = 1$.

V. IMPROVED BOUNDS ON GROUND-STATE CORRELATION DECAY

In this section, we show that the method in Sec. II A also significantly improves bounds on the correlation decay of gapped (possibly degenerate) ground states, compared to previous results [2,40]. We first obtain an integral formula that relates $\Omega_{XY}(\omega)$ in Eq. (9) and the connected correlation function $\langle \hat{S}_X \hat{V}_Y \rangle_c \equiv \langle \hat{S}_X \hat{V}_Y \rangle - \langle \hat{S}_X \rangle \langle \hat{V}_Y \rangle$ in the gapped ground state $|G\rangle$. Integrating Eq. (9) along the imaginary axis, we have

$$\begin{aligned}\int_{-\infty i}^{+\infty i} \Omega_{XY}(\omega) d\omega &= \int_{-\infty i}^{+\infty i} d\omega \langle G|\hat{S}_X \frac{i\bar{P}_G}{\omega - \hat{\Delta}} \hat{V}_Y|G\rangle \\ &\quad - \int_{-\infty i}^{+\infty i} d\omega \langle G|\hat{V}_Y \frac{i\bar{P}_G}{\omega + \hat{\Delta}} \hat{S}_X|G\rangle \\ &= \pi \langle G|\hat{S}_X \bar{P}_G \hat{V}_Y|G\rangle + \text{c.c.} \\ &= 2\pi \langle \hat{S}_X \hat{V}_Y\rangle_c, \end{aligned} \tag{33}$$

where we use the following equality:

$$\int_{-\infty i}^{+\infty i} \frac{1}{\omega - \mu} d\omega = -\pi i \,\mathrm{sgn}(\mu). \tag{34}$$

With Eq. (33), we can obtain an upper bound on $|\langle \hat{S}_X \hat{V}_Y\rangle_c|$ by integrating $|\Omega_{XY}(\omega)|$ along the imaginary axis. Furthermore, it can be proved that (see Appendix D) $|\Omega_{XY}(\omega)|$ on the imaginary axis can always be upper bounded by the upper bound of $|\Omega_{XY}(0)|$ obtained by Eq. (13) (we denote this upper bound by $|\bar{\Omega}_{XY}(0)|$). Therefore, we can use the upper bound $|\Omega_{XY}(iy)| \le \min[|\bar{\Omega}_{XY}(0)|, \bar{\Omega}(d_{XY}, y)]$. Note that the integration on this bound on iy is guaranteed to converge provided that we use the best LRB, since $C(d_{XY}, t) \propto t^{\nu}$ at small t with $\nu \ge 1$ and so $\bar{\Omega}(d_{XY}, y)$ in Eq. (11) decays at least as $y^{-\nu-1}$ at large y. This upper bound yields

$$\begin{aligned} 2\pi|\langle \hat{S}_X \hat{V}_Y\rangle_c| &\le \int_{-\infty}^{+\infty} |\Omega_{XY}(iy)| dy \\ &\le 2y_0|\bar{\Omega}_{XY}(0)| + 2\int_{y_0}^{\infty} \bar{\Omega}(d_{XY}, y) dy, \end{aligned} \tag{35}$$

for any $y_0 > 0$ [for the optimal result, y_0 should satisfy $\bar{\Omega}(d_{XY}, y_0) = |\bar{\Omega}_{XY}(0)|$].

For example, for $D < \alpha < 2D$, we have

$$\bar{\Omega}(r, y) \le \frac{\mathbf{C}}{r^{\alpha}} \left[\frac{e^{(\upsilon - y)t_0} - 1}{\upsilon - y} + \frac{e^{-yt_0} - 1}{y} \right] + \mathbf{C}\frac{e^{-yt_0}}{y}, \tag{36}$$

which is obtained by computing the first line of Eq. (18) exactly without using any simplifications. Inserting Eq. (36) into Eq. (35) and taking $y_0 = \upsilon$, we see that the integral of the term in square brackets converges to a constant independent of r and therefore the second term in the last line of Eq. (35) is bounded by $\mathbf{C}/d_{XY}^{\alpha}$. For $|\bar{\Omega}_{XY}(0)|$, we use the results of Appendix B 3 [Eqs. (B10) and (B11)]. Eventually, we obtain

$$|\langle \hat{S}_X \hat{V}_Y\rangle_c| \le \mathbf{P}(\ln d_{XY})/d_{XY}^{\alpha_1}, \tag{37}$$

where $\mathbf{P}(x)$ is a quadratic polynomial in x. Other cases in Table I can be treated in an identical manner, by inserting the results of Sec. II into Eq. (35). In all cases, we obtain Eq. (4) with $\alpha_2 = \alpha_1$ for the power-law cases or $\mu_2 = \mu_1$ for short-range interacting cases.

VI. CONCLUSIONS

We prove a locality principle for gapped ground states in systems with power-law- $(1/r^{\alpha})$ decaying interactions: when $\alpha > D$, the response of a local observable $\hat{S}_X$ to a spatially separated local perturbation $\hat{V}_Y$ decays as a power law $(1/r^{\alpha_1})$ in distance, provided that $\hat{V}_Y$ does not close the spectral gap. When $\alpha > 2D$, the bound on the exponent α_1 that we obtain, $\alpha_1 = \alpha$, is tight. We prove this using a method that avoids the use of QAC and incorporates techniques of complex analysis. Our method also improves bounds on ground-state correlation decay, even in short-range interacting systems.

Our results have profound significance in studying the ground-state properties of power-law-interacting systems. At a fundamental level, the LPPL bounds generalize the notion of locality to gapped ground states of power-law systems, implying that the local properties of such ground states are stable against distant local perturbations. At a more practical level, we show how our results immediately lead to an upper bound on the FSE in numerical simulations of gapped ground states, which reveals that FSEs generally decay as a power law $(1/L^{\alpha_3})$ in system size (provided that α or the spectral gap Δ is not too small). A corollary of this is the existence of thermodynamic limit for local observables in ground states of power-law systems, under the spectral gap assumption stated in Sec. IV.

We now discuss some open questions and future directions. One open question concerns whether the power-law exponents α_1 and α_2 given in Table I are tight when $D < \alpha < 2D$: we see that in this case both of them are strictly smaller than α, yet for all gapped power-law systems that we know, no correlations decay slower than $1/r^{\alpha}$, which strongly suggests that our bounds can further be improved in this case. An interesting future direction is to generalize our results to systems of interacting bosons, such as the Bose-Hubbard model, where our current bounds do not apply due to the interaction $\hat{h}_X$ in Eq. (6) having infinite norm, thereby violating Eq. (16) and the corresponding LRBs. However, our method in Sec. II A still works if we incorporate Eq. (11) with recent LR-type bounds for interacting bosons [3,55–57]. It will then be interesting to see how the exponents in Table I get modified. Another future direction is to prove the stability of the spectral gap against extensive local perturbations in gapped frustration-free ground states of power-law Hamiltonians. For locally interacting systems, this has been proved under the local topological quantum order condition [58–60], where an essential tool in the proof is Hastings' QAC [Eq. (2)]. It is interesting to investigate if our new method can improve these results and extend them to power-law systems.

ACKNOWLEDGMENTS

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APPENDIX A: LPPL BOUND FROM QAC

In this appendix, we briefly show how to obtain an LPPL bound by directly generalizing the previous method based on QAC. We will see that QAC only leads to an LPPL bound for $\alpha > 2D$, where it gives $\alpha_1 = \alpha - D - 1$, much looser than our bound $\alpha_1 = \alpha$ (which we know to be tight).

We use the same setup as in Sec. II. QAC constructs a unitary evolution process relating the ground states of different λ,

$$i\partial_\lambda|G(\lambda)\rangle = \hat{D}(\lambda)|G(\lambda)\rangle, \tag{A1}$$

where $\hat{D}(\lambda)$ is a Hermitian operator that depends on $\hat{H}(\lambda)$. Following the derivations in Ref. [38, Eqs. (5)–(9)], $\hat{D}(\lambda)$ can be expanded as

$$\hat{D}(\lambda) = \sum_{R=1}^{\infty} \hat{V}_Y(\lambda, R), \tag{A2}$$

where $\hat{V}_Y(\lambda, R)$ is an operator that acts only on sites within a distance R from Y. For $\alpha > 2D$, $\|\hat{V}_Y(\lambda, R)\| \leq \mathbf{C}\|\hat{V}_Y\|/R^{\alpha-D}$, while for $\alpha \leq 2D$, $\|\hat{V}_Y(\lambda, R)\|$ decays more slowly than any power in R [38]. We now use the previous method [5,9] to bound $|\delta\langle\hat{S}_X\rangle_{\hat{V}_Y}|$:

$$\begin{aligned} |\delta\langle\hat{S}_X\rangle_{\hat{V}_Y}| &\equiv |\langle\hat{S}_X\rangle_{\lambda=1} - \langle\hat{S}_X\rangle_{\lambda=0}| \\ &\leq \int_0^1 |\partial_\lambda\langle\hat{S}_X\rangle_\lambda| d\lambda \\ &= \int_0^1 |\langle[\hat{S}_X, \hat{D}(\lambda)]\rangle_\lambda| d\lambda \\ &\leq \int_0^1 \|[\hat{S}_X, \hat{D}(\lambda)]\| d\lambda. \end{aligned} \tag{A3}$$

For $\alpha > 2D$, the integrand is bounded as

$$\begin{aligned} \|[\hat{S}_X, \hat{D}(\lambda)]\| &= \sum_{R\geq d_{XY}} \|[\hat{S}_X, \hat{V}_Y(\lambda, R)]\| \\ &\leq \sum_{R\geq d_{XY}} \mathbf{C}\frac{1}{R^{\alpha-D}} \\ &= \mathbf{C}\frac{1}{d_{XY}^{\alpha-D-1}}. \end{aligned} \tag{A4}$$

This leads to the bound $|\delta\langle\hat{S}_X\rangle_{\hat{V}_Y}| \leq \mathbf{C}/d_{XY}^{\alpha-D-1}$ for $\alpha > 2D$, while for $\alpha \leq 2D$, the bound obtained this way decays more slowly than any power in d_{XY}, verifying our earlier claims.

APPENDIX B: SOME DETAILS FOR SEC. II

In this appendix, we provide some technical details for Sec. II.

1. Details for Sec. II B

We first briefly explain how Eq. (14) is obtained from Ref. [31]. The main result of Ref. [31] is stated in their Eq. (18):

$$C(r,t) \leq \mathbf{C}\exp\left(vt - \frac{r}{\chi}\right) + \mathbf{C}\frac{e^{v_\chi t}}{[r/R(t)]^\alpha}, \tag{B1}$$

valid when $vt > \alpha\ln\alpha$ and $r > 6R(t)$, where $R(t) = \chi vt$, $v_\chi = \mathbf{C}R(t)^D\lambda_\chi$, and $\lambda_\chi = \sup_{i\in\Lambda}\sum_{j:d_{ij}\geq\chi}\|\hat{h}_{ij}\|$. For $\chi > \mathbf{C}$, we have $\lambda_\chi \leq \mathbf{C}\chi^{D-\alpha}$. We now take $\chi = C_0t^\gamma$ where $C_0 > 0$ is a constant, so $R(t) = C_0vt^{\gamma+1}$ and $r > 6R(t)$ is equivalent to $t < (r/6vC_0)^{1/(\gamma+1)} \equiv t_0$. For $t > (\alpha\ln\alpha)/v$, we have $\chi > \mathbf{C}$, leading to $v_\chi t \leq \mathbf{C}$. Inserting $R(t)$ and χ into Eq. (B1), we obtain Eq. (14). By taking the second derivative (with respect to t) of the first term in the rhs of Eq. (B1), we see that this term is indeed convex provided that the constant C_0 is chosen to be large enough, verifying our claim below Eq. (15).

We now insert Eq. (15) into Eq. (13) to prove Eq. (3). We first simplify the last line of Eq. (15): note that for $y = |\mathrm{Im}[\omega]| = \rho|\sin\theta| \leq \rho$, we have $e^{(v-y)t_0 - r/(C_0t_0^\gamma)} \leq \mathbf{C}e^{-yt_0}/y$, $r^{-\alpha} \leq \mathbf{C}r^{-\alpha}y^{-\alpha(\gamma+1)-1}$ and $t_0e^{-\mathbf{C}r} \leq \mathbf{C}r^{-\alpha}y^{-\alpha(\gamma+1)-1}$ (for $r \geq 1$). Therefore,

$$\bar{\Omega}(r,y) \leq (\mathbf{C}t_0 + \mathbf{C}y^{-1})e^{-yt_0} + \mathbf{C}r^{-\alpha}y^{-\alpha(\gamma+1)-1}. \tag{B2}$$

The second term in Eq. (B2) dominates at small and large y, while the first term is only important in an intermediate region (y_1, y_2), where $y_{1,2} = x_{1,2}r^{-1/(\gamma+1)}$ and x_1, x_2 are the two solutions to the equation (and are independent of r):

$$(x + \mathbf{C})e^{-\mathbf{C}x} = x^{-\alpha(\gamma+1)}. \tag{B3}$$

In summary,

$$\bar{\Omega}(r, y) \le \begin{cases} (\mathbf{C}t_0 + \mathbf{C}y^{-1})e^{-yt_0}, & y_1 \le y \le y_2, \\ \mathbf{C}r^{-\alpha}y^{-\alpha(\gamma+1)-1}, & 0 < y < y_1 \text{ or } y > y_2. \end{cases} \tag{B4}$$

[In the event that Eq. (B3) has no solution, then $\bar{\Omega}(r, y)$ is always bounded by the second line of Eq. (B4) and our following derivations still work with minor modifications.] Inserting Eq. (B4) into Eq. (13), we have

$$\ln|\Omega_{XY}(0)| \le \ln\mathbf{C} - \frac{2\alpha}{\pi}(\pi/2 - \theta_2 + \theta_1)\ln r + \frac{2}{\pi}\int_{\theta_1}^{\theta_2}\left[\ln\left(\mathbf{C}t_0 + \frac{\mathbf{C}}{\sin\theta}\right) - t_0\rho\sin\theta\right]d\theta, \tag{B5}$$

where $y_{1,2} \equiv \rho\sin\theta_{1,2} = x_{1,2}r^{-1/(\gamma+1)}$. Using $\theta_{1,2} = O[r^{-1/(\gamma+1)}]$, we see that all but the $\ln\mathbf{C} - \alpha\ln r$ term are of order $r^{-1/(\gamma+1)}$, $r^{-1/(\gamma+1)}\ln r$, or $r^{-2/(\gamma+1)}$, all of which are upper bounded by a constant for $r \ge 1$. This proves Eq. (3), with the subleading factor $\mathbf{P}(\ln r)$ being a constant.

2. Improving the bound on $|\Omega_{XY}(0)|$ with conformal mapping

We now introduce a technique to further improve the bound in Eq. (13), which leads to an improvement of the bound on α_1 in Sec. II C. The basic idea is to apply a suitable conformal mapping to $\Omega_{XY}(\xi)$ before applying the bound Eq. (13). To be specific, let $f(\xi)$ be a complex analytic function in the open unit disk D, such that $f(0) = 0$ and $f(D) \cap K_\Delta = \emptyset$. Then $\Omega_{XY}[f(\xi)]$ is complex analytic for $\xi \in D$, so according to Lemma 1, $\ln|\Omega_{XY}[f(\xi)]|$ is subharmonic in D and therefore for any $\rho \in (0, 1)$,

$$\ln|\Omega_{XY}(0)| \le \frac{1}{2\pi}\int_0^{2\pi}\ln|\Omega_{XY}[f(\rho e^{i\theta})]|d\theta \le \frac{1}{2\pi}\int_0^{2\pi}\ln\bar{\Omega}[d_{XY}, |\mathrm{Im} f(\rho e^{i\theta})|]d\theta. \tag{B6}$$

Note that Eq. (13) in Sec. II A corresponds to the special case $f(\xi) = \Delta\xi$. Since the inequality given in Eq. (B6) holds for all such functions $f(\xi)$ (satisfying the conditions mentioned above), we can choose a $f(\xi)$ to optimize this bound. We show in the next section how this additional conformal mapping improves the bound in Sec. II C.

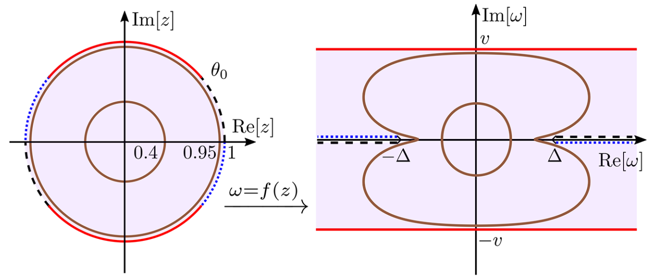

FIG. 2. For any finite system size L, K_Δ (dashed region on the real axis of the right panel) contains all possible pole positions of the rhs of Eq. (9), so $\Omega_{XY}(\omega)$ is complex analytic in the region $\mathbb{C}\backslash K_\Delta$. The conformal mapping $\omega = f(z)$ defined in Eq. (B7) maps the unit disk (left) to the shaded region of the infinite strip with the pole regions excluded (right).

3. Improving the bound in Sec. II C

We begin by inserting Eq. (18) into Eq. (B6), with the conformal mapping [61]

$$f(z) \equiv \frac{2v}{\pi}\,\mathrm{arctanh}\left(\frac{2z}{z^2+1}\tanh\frac{\Delta\pi}{2v}\right). \tag{B7}$$

The image of the unit disk under the mapping $\omega = f(z)$ is shown in Fig. 2. Note that $y(\theta) \equiv |\mathrm{Im}[f(\rho e^{i\theta})]| < v$ for $\rho \in (0, 1), \theta \in [0, 2\pi]$, so Eq. (B6) becomes

$$\ln|\Omega_{XY}(0)| \le \frac{2}{\pi}\int_0^{\pi/2}[\ln(\mathbf{C}t_0) - \ln y(\theta) - y(\theta)t_0]d\theta = \ln(\mathbf{C}t_0) - \frac{2}{\pi}\int_0^{\pi/2}[\ln y(\theta) + y(\theta)t_0]d\theta. \tag{B8}$$

Before going into more technical calculations, we first give some heuristic arguments about the asymptotic behavior of the $|\Omega_{XY}(0)|$ at large r and guess the exponent α_1. We will see later that the asymptotic behavior of the last line of Eq. (B8) at large r is dominated by the third term, since the first two terms have much weaker dependence on r. The third term in the last line of Eq. (B8) decreases as ρ gets closer to 1 and in the limit $\rho \to 1$, $y(\theta)$ becomes a step function: $y(\theta) = 0$ for $\theta < \theta_0$ while $y(\theta) = v$ for $\theta > \theta_0$, where θ_0 satisfies $\cos\theta_0 = \tanh(\Delta\pi/2v)$ and is marked in Fig. 2. Therefore, in the limit $\rho \to 1$, the third term in the last line of Eq. (B8) is

$$-t_0\frac{2}{\pi}\int_0^{\pi/2}y\,d\theta = -2t_0v(\pi/2 - \theta_0)/\pi = -\frac{2t_0v}{\pi}\arcsin\left(\tanh\frac{\Delta\pi}{2v}\right). \tag{B9}$$

The subtlety here is that the first two terms in the last line of Eq. (B8) diverge as $\rho \to 1$. In the following, we show that by choosing ρ suitably close to 1, we can obtain a bound

$$|\Omega_{XY}(0)| \leq P(\ln r) r^{-\alpha_1}, \tag{B10}$$

where $\mathbf{P}(x)$ is a quadratic polynomial in x and

$$\alpha_1 = \frac{2\alpha}{\pi} \arcsin\left(\tanh\frac{\Delta\pi}{2v}\right). \tag{B11}$$

We begin by upper bounding the first term in the integrand in Eq. (B8). Due to the symmetry $y(\theta) = y(\pi - \theta) = y(\pi + \theta)$, we only need to treat the integrand in the interval $\theta \in [0, \pi/2]$. To this end, we obtain a simple lower bound for $y(\theta)$ as follows:

$$\begin{aligned} y(\theta) &= \frac{2v}{\pi} \mathrm{Im}\left[\operatorname{arctanh}\left(\frac{2z}{z^2+1} \tanh\frac{\Delta\pi}{2v}\right)\right] \\ &\geq \frac{2v}{\pi} \arctan\left[\mathrm{Im}\left(\frac{2z}{z^2+1}\right) \tanh\frac{\Delta\pi}{2v}\right] \\ &= \mathbf{C} \arctan\left[\mathbf{C}\frac{(1-\rho^2)\rho\sin\theta}{\rho^4 + 2\rho^2\cos 2\theta + 1}\right] \\ &\geq \mathbf{C}(1-\rho)\sin\theta, \end{aligned} \tag{B12}$$

for $\rho \geq 0.9$ and $\theta \in [0, \pi/2]$, where in the second line we use $\mathrm{Im}[\operatorname{arctanh}(z)] \geq \arctan \mathrm{Im}[z]$ for z in the upper half plane (which follows from the fact that $\mathrm{Im}[\operatorname{arctanh}(x + i\epsilon)]$ is monotonically increasing in x for $\epsilon > 0, x > 0$) and the proof for the last line is elementary. Therefore, the second term in the last line of Eq. (B8) can be upper bounded by

$$-\frac{2}{\pi}\int_0^{\pi/2} \ln y(\theta) d\theta \leq \ln \mathbf{C} - \ln(1-\rho). \tag{B13}$$

This may be a crude bound but it captures the leading singularity of this term as $\rho \to 1$. We now study the second term in the integrand in Eq. (B8) near $\rho \to 1$. We have

$$\begin{aligned} \partial_\rho y(\theta) &= \mathrm{Im}[\partial_\rho f(z)] \\ &= \mathrm{Im}\left[\frac{1}{i\rho}\partial_\theta f(z)\right] \\ &= -\frac{1}{\rho}\mathrm{Re}[\partial_\theta f(z)] \end{aligned} \tag{B14}$$

and therefore

$$\begin{aligned} \partial_\rho \int_0^{\pi/2} y(\theta) d\theta &= -\frac{1}{\rho}\int_0^{\pi/2} d\theta\, \partial_\theta \mathrm{Re}[f(z)] \\ &= -\frac{1}{\rho}\mathrm{Re}[f(i\rho) - f(\rho)] \\ &= \frac{1}{\rho} f(\rho), \end{aligned} \tag{B15}$$

the limit of which at $\rho \to 1$ is Δ. Since this derivative exists for $\rho \in [\rho_0, 1]$ for any $\rho_0 > 0$, along with Eq. (B9), we obtain

$$\int_0^{\pi/2} y d\theta \geq C_0 - \mathbf{C}(1-\rho), \tag{B16}$$

where $C_0 \equiv v \arcsin[\tanh(\Delta\pi/2v)]$. Inserting Eqs. (B13) and (B16) into Eq. (B8), we obtain

$$\ln|\Omega_{XY}(0)| \leq -\alpha_1 \ln r + \ln\frac{\mathbf{C}t_0}{1-\rho} + \mathbf{C}t_0(1-\rho). \tag{B17}$$

Minimizing the rhs of Eq. (B17) [the minimum is at $1 - \rho = 1/(\mathbf{C}t_0)$], we obtain Eq. (B10) where the polynomial prefactor can be taken as $P(\ln r) = \mathbf{C}(\ln r)^2$.

Comparing Eqs. (B11) and (20), we see that the technique here improves α_1 at all values of Δ/v, especially when Δ/v is large, where α_1 approaches α exponentially fast in Eq. (B11), while $\alpha - \alpha_1 \propto v/\Delta$ in Eq. (20).

APPENDIX C: FSE BOUNDS

In this appendix, we provide some missing details in Sec. IV, including a proof of Lemma 2 and a derivation of the bounds in arbitrary spatial dimension.

1. Proof of Lemma 2

For simplicity, we assume that $R = 2R_1 + 1$ is an odd number (the proof for even R is similar). We have

$$\begin{aligned} \sum_{r=1}^{R-1} \frac{\mathbf{P}(\ln r)}{r^\zeta (R-r)^\eta} &= \left(\sum_{r=1}^{R_1} + \sum_{r=R_1+1}^{R-1}\right) \frac{\mathbf{P}(\ln r)}{r^\zeta (R-r)^\eta} \\ &= \sum_{r=1}^{R_1} \left\{\frac{\mathbf{P}(\ln r)}{r^\zeta (R-r)^\eta} + \frac{\mathbf{P}[\ln(R-r)]}{r^\eta (R-r)^\zeta}\right\} \\ &\asymp \sum_{r=1}^{R_1} \left[\frac{\mathbf{P}(\ln r)}{r^\zeta R^\eta} + \frac{\mathbf{P}(\ln R)}{r^\eta R^\zeta}\right], \\ &\leq \sum_{r=1}^{R_1} \left[\frac{\mathbf{P}(\ln R)}{r^\zeta R^\eta} + \frac{\mathbf{P}(\ln R)}{r^\eta R^\zeta}\right], \end{aligned} \tag{C1}$$

where in the second line we substitute r by $R - r$ in the second sum and in the third line we use $\mathbf{P}[\ln(R-r)](R-r)^{-\gamma} \asymp \mathbf{P}(\ln R) R^{-\gamma}$ for $1 \leq r \leq R_1$ and $\gamma > 0$, since $\mathbf{P}[\ln(R/2)] \leq \mathbf{P}[\ln(R-r)] \leq \mathbf{P}(\ln R)$ and $R^{-\gamma} \leq (R-r)^{-\gamma} \leq (R/2)^{-\gamma}$. Now applying $\sum_{r=1}^{R_1} r^{-\gamma} \asymp \int_{r=1}^{R_1} r^{-\gamma} dr$ to the last line of Eq. (C1) and calculating the integral, we obtain Eq. (32). Note that the $\mathbf{P}(x)$ in the rhs of Eq. (32) may be higher in degree (higher by at most 1) than the $\mathbf{P}(x)$ in the lhs, since the summation $\sum_{r=1}^{R_1} r^{-\gamma}$ introduces an additional $\ln R$ factor when $\gamma = 1$.

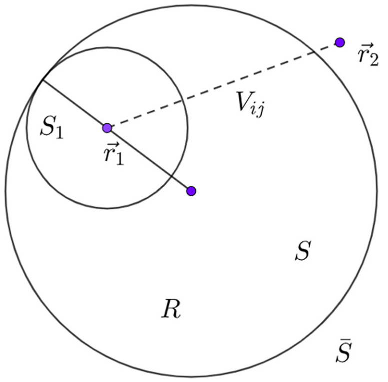

FIG. 3. The derivation of the FSE bound in higher spatial dimension. S is the finite system with radius R, $\overline{S}$ is its complement, $\vec{r}_1$ and $\vec{r}_2$ are, respectively, the positions i and j of the power-law interaction V_{ij}, and $S_1 \subseteq S$ is a subsystem centered at $\vec{r}_1$ with radius $R - r_1$ (so that S_1 touches S).

2. Derivation of the bounds in higher dimension

In Sec. IV, we derive the FSE bound Eq. (5) in 1D. In the following, we generalize the derivation to arbitrary spatial dimension. The configuration is shown in Fig. 3. Without loss of generality, we can assume that the system has a spherical shape (the sphere S in Fig. 3), since the error in other cluster shapes can be upper and lower bounded by spheres with radii proportional to their linear dimensions. Equation (30) is still valid, so we have

$$\begin{aligned}|\delta\langle\hat{S}_X\rangle_L| &\leq \sum_{\vec{r}_1\in S,\vec{r}_2\in\overline{S}} \frac{\mathbf{P}(\ln r_1)}{r_1^{\alpha_1}|\vec{r}_2-\vec{r}_1|^{\alpha}} \\ &\leq \sum_{\vec{r}_1\in S,\vec{r}_2\in\overline{S_1}} \frac{\mathbf{P}(\ln r_1)}{r_1^{\alpha_1}|\vec{r}_2-\vec{r}_1|^{\alpha}} \\ &\leq \sum_{\vec{r}_1\in S} \frac{\mathbf{P}(\ln r_1)}{r_1^{\alpha_1}(R-r_1)^{\alpha-D}} \\ &\leq \sum_{r_1=1}^{R-1} \frac{\mathbf{P}(\ln r_1)}{r_1^{\alpha_1-D+1}(R-r_1)^{\alpha-D}},\end{aligned} \tag{C2}$$

where in the second line we extend the sum in $\vec{r}_2$ from $\overline{S}$ to $\overline{S_1}$ (which contains $\overline{S}$), in the third and the last lines, we upper bound the sums by integration [with a constant coefficient absorbed into $\mathbf{P}(\ln r_1)$] and the integrals can be calculated analytically due to the spherical geometry. Applying Lemma 2 to Eq. (C2), we obtain Eq. (5), with α_3 summarized in Table I.

APPENDIX D: BOUNDS FOR CORRELATION DECAY: PROOF THAT $|\Omega_{XY}(iy)| \leq |\bar{\Omega}_{XY}(0)|$

In this appendix, we prove the claim we make in Sec. V that $|\Omega_{XY}(iy)|$ for $y \in \mathbb{R}$ can always be upper bounded by the upper bound of $|\Omega_{XY}(0)|$ obtained by Eq. (13). We prove this for the $\Omega_{XY}(\omega)$ in Sec. II C and Appendix B 3, i.e., power-law systems with $\alpha > D$, and the proofs for other cases are similar.

We begin by recalling a simple fact about subharmonic functions: if $p(\omega)$ is a real-valued subharmonic function and $q(\omega)$ is a real-valued harmonic function such that $p(\omega) \leq q(\omega)$ on the boundary of a simply connected domain S, then $p(\omega) \leq q(\omega)$ everywhere in S. Now take $p(\omega)$ to be the subharmonic function $\ln|\Omega_{XY}(\omega)|$ and take $q(\omega)$ to be the unique harmonic function that agrees with $\ln\bar{\Omega}(r,y)$ on the boundary of the region S_ρ bounded by the parametric curve $f(\rho e^{i\theta}), \theta \in [0, 2\pi]$, as plotted in Fig. 2, where $f(z)$ is defined in Eq. (B7), $\rho \in (0,1)$, and later we consider the limit $\rho \to 1$. By construction, we have $p(\omega) \leq q(\omega)$ on ∂S_ρ; therefore, $p(\omega) \leq q(\omega)$ everywhere in S_ρ. Using the mean-value property of harmonic functions, in the limit $\rho \to 1$ we have

$$\begin{aligned}\lim_{\rho\to 1} q(0) &= \lim_{\rho\to 1}\frac{1}{2\pi}\int_0^{2\pi} q[f(\rho e^{i\theta})]d\theta \\ &= \lim_{\rho\to 1}\frac{1}{2\pi}\int_0^{2\pi} \ln\bar{\Omega}[r, |\mathrm{Im} f(\rho e^{i\theta})|]d\theta \\ &= \ln|\bar{\Omega}_{XY}(0)|.\end{aligned} \tag{D1}$$

Therefore, to prove that $|\Omega_{XY}(iy)| \leq |\bar{\Omega}_{XY}(0)|$, it suffices to prove that $q(iy)$ is monotonically decreasing in y for $y \geq 0$. In the following, we prove this for any $\rho \in (0,1)$.

Since $q(\omega)$ is harmonic, for illustrative purposes we use the language of electrostatics. From the expression of $\bar{\Omega}(r,y)$ in Eq. (18) it is clear that on the boundary of S_ρ the potential $q(\omega)$ is strictly decreasing in the direction of increasing $|y|$. In the following, we use proof by contradiction: if $q(iy)$ is not monotonic in y for $y \geq 0$, then there must exist y_1, y_2 satisfying $0 < y_1 < y_2 < f(i\rho)/i$ such that $q(iy_1) = q(iy_2)$. Let l_1, l_2 be the equipotential lines passing through iy_1 and iy_2, respectively. Equipotential lines cannot terminate in free space, since otherwise this would imply that there is an electric charge at the end point. l_1 and l_2 cannot intersect anywhere, since, for example, if they intersect at a point $x + iy$ with $x > 0, y > 0$, then by symmetry they also intersect at $-x + iy$, which implies that l_1 and l_2 enclose a region in which $q(\omega)$ is a constant, which is impossible for a nonconstant harmonic function. By similar logic (and using the mirror symmetry with respect to the real axis), neither l_1 nor l_2 can intersect with the real axis, so we can focus our attention on the upper half plane. Furthermore, at most one of l_1 and l_2 can intersect with the boundary of S_ρ, since $q(\omega)$ is strictly decreasing in the direction of increasing $|y|$ on the boundary. Without loss of generality, suppose that l_1 does not intersect the boundary. Then the only remaining possibility is that l_1 is a closed curve inside S_ρ. But this implies that $q(\omega)$ is constant in the interior of l_1, which is impossible for a nonconstant harmonic function. In conclusion, $q(iy)$ must be monotonically decreasing in y for $y > 0$. [It is straightforward to rule out the possibility of $q(iy)$ being monotonically increasing in y: since if that is the case, there must exist x, y with $0 < x < f(\rho), 0 < y < f(i\rho)/i$ such that $q(x) = q(iy)$. Considering the equipotential curve passing through x, iy, $-x$, and $-iy$, we reach a similar contradiction.]

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[47] For fermionic systems, $\hat{h}_X$ is an operator that is even in the fermion creation and annihilation operators and only involves fermionic modes inside the region X. This is enough to guarantee the important condition for the LRBs used later in this paper: for nonoverlapping regions X_1, X_2 (i.e., $X_1 \cap X_2 = \emptyset$), we always have $[\hat{h}_{X_1}, \hat{h}_{X_2}] = 0$.

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[51] Note that this definition of two-body interactions is more general than the more common definition—here, we actually allow interaction between any two clusters of sites, where the radius of each cluster is no larger than a fixed number. It is straightforward to see that the results in Ref. [31] are still valid for two-cluster interactions, since one can always regroup lattice sites to make interactions two-body, as long as the cluster sizes have an upper bound.

[52] Although for the case $\alpha > 2D + 1$, stronger LRBs with linear light-cone have been obtained in Refs. [33,35], those bounds do not lead to qualitatively tighter LPPL bounds. Indeed, the LRB in Ref. [33] gives $C(r,t) \le t/r$ in 1D, leading to an LPPL bound with $\alpha_1 = 1$, which is worse than our current results, and the LRB in Ref. [35] can at most improve the subleading prefactor in Eq. (3), since $\alpha_1 = \alpha$ is already tight for generic systems.

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