Quantum Nonreciprocal Interactions via Dissipative Gauge Symmetry

Yu-Xin Wang,[1,*] Chen Wang,[2] and Aashish A. Clerk[1]

[1] *Pritzker School of Molecular Engineering, University of Chicago, Chicago, Illinois 60637, USA*
[2] *Department of Physics, University of Massachusetts-Amherst, Amherst, Massachusetts 01003, USA*

(Received 4 April 2022; accepted 16 December 2022; published 17 January 2023)

One-way nonreciprocal interactions between two quantum systems are typically described by a cascaded quantum master equation, and rely on an effective breaking of time-reversal symmetry (TRS) as well as the balancing of coherent and dissipative interactions. Here, we present a new approach for obtaining nonreciprocal quantum interactions that is *completely distinct* from cascaded quantum systems, and that does not in general require broken TRS. Our method relies on a local gauge symmetry present in any Markovian Lindblad master equation. This new kind of quantum nonreciprocity has many implications, including a new mechanism for performing dissipative steady-state unitary gate operations on a target quantum system. We also introduce a new, extremely general quantum-information-based metric for quantifying quantum nonreciprocity.

DOI: 10.1103/PRXQuantum.4.010306

I. INTRODUCTION

The study of interactions and scattering that are intrinsically directional (i.e., nonreciprocal) is at the forefront of many areas of physics. Such interactions are of fundamental interest: for example, they can lead to exotic phase transitions in classical active matter systems [1–3], and can also be used to generate dimerized many-body entangled states [4–6]. They also have a myriad of practical applications in both classical and quantum information processing tasks, in settings that range from classical photonic and acoustic systems [7–9] to quantum circuits and networks [10–19].

While classically, one can describe directional interactions using effective non-Hermitian Hamiltonians, in quantum settings one needs a description that conserves probability and accounts for quantum fluctuations. The standard quantum description of nonreciprocity is provided by the theory of cascaded quantum systems [20,21]. It describes an extremely general class of fully directional interactions between two subsystems A and B that involve a pair of arbitrary "local" operators $\hat{A}$ and $\hat{B}$ (i.e., $\hat{A}$ only acts on subsystem A, $\hat{B}$ only acts on subsystem B). The directional dynamics is described by a Lindblad quantum master equation (QME) of the form [20–22]

$$\frac{d\hat{\rho}}{dt} = -i[\hat{H}_{AB}, \hat{\rho}] + \mathcal{D}[\hat{A} - i\hat{B}]\hat{\rho} \equiv \mathcal{L}_{\rm CS}\hat{\rho}, \qquad (1)$$

where $\hat{\rho}$ is the system density matrix, $\hat{H}_{AB} = (\hat{A}^\dagger\hat{B} + \hat{B}^\dagger\hat{A})/2$, and $\mathcal{D}[\hat{O}]\hat{\rho} = (\hat{O}\hat{\rho}\hat{O}^\dagger - \{\hat{O}^\dagger\hat{O}, \hat{\rho}\}/2)$ denotes the standard Lindblad dissipator. One can show that the dynamics encodes a fully one-way interaction where subsystem A affects the dynamics and evolution of subsystem B, but not vice versa.

Cascaded QMEs were first derived for setups involving an explicitly nonreciprocal element (e.g., a directional waveguide or circulator). More recently, it was realized that Eq. (1) provides a more general blueprint for engineering nonreciprocal interactions, based on balancing a coherent Hamiltonian interaction (described by $\hat{H}_{AB}$) and a dissipative interaction (described by the dissipator $\mathcal{D}[\hat{A} - i\hat{B}]\hat{\rho}$) [22,23]. This can be realized by engineering suitable drives and couplings to dissipative environments, without using a conventional nonreciprocal element. This approach has been employed in a variety of experiments, including quantum optomechanics (e.g., Refs. [13,15–18]) and superconducting quantum circuits (e.g., Refs. [12,14]). Note that the coefficient $-i$ in the dissipator of Eq. (1) ultimately implies that any physical means for realizing this dynamics requires an effective broken time-reversal symmetry (TRS). This can be achieved by using phases encoded in the drive tones applied to the system, in a manner that generates a synthetic gauge flux. This time-modulation approach to nonreciprocity is also well studied in completely classical contexts [7–9].

*yxwang@uchicago.edu

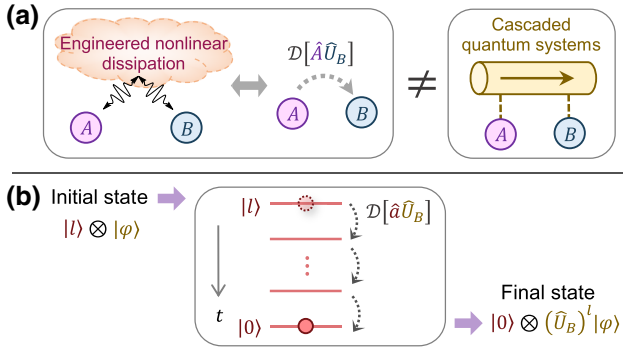

FIG. 1. (a) Schematic for quantum nonreciprocal interactions via gauge symmetry. This approach uses correlated dissipation, but is completely distinct from a cascaded quantum system. (b) Dissipative steady-state realization of a tunable unitary gate making use of the nonreciprocal interaction in (a), in a system where subsystem A is a cavity mode a that controls the gate, and subsystem B is the target qubit. One can selectively apply a unitary gate $(\hat{U}_B)^\ell$ on B by initializing A in the corresponding Fock state $|\ell\rangle$, and letting the system relax to the steady state [see Eq. (3) and Sec. III C].

One might guess that the general structure of Eq. (1) is the only way to obtain fully directional, Markovian quantum interactions between two systems. In this work we show that this is not the case. We introduce a new kind of quantum open-system dynamics where correlated dissipation generates nonreciprocal interactions in a manner distinct from the cascaded QME. As we discuss in detail, we ultimately exploit a basic gauge symmetry present in any Lindblad master equation. This gives us a mechanism for nonreciprocity that, surprisingly, does not require any notion of broken TRS or a nontrivial synthetic gauge field. In its simplest form, the new nonreciprocal QME can be written in terms of a generic local operator $\hat{A}$ on A and a unitary $\hat{U}_B$ on B, as [see Fig. 1(a)]

$$\frac{d\hat{\rho}}{dt} = \Gamma\mathcal{D}[\hat{A}\hat{U}_B]\hat{\rho} \equiv \mathcal{L}_{\rm dir}\hat{\rho}. \tag{2}$$

As we show, this purely dissipative dynamics is strictly unidirectional (A influences B but not vice versa), and, moreover, cannot be written in the cascaded QME form of Eq. (1). We also show that this structure has a nontrivial generalization to more complex master equations with multiple dissipators.

Our work provides a thorough investigation of this new route to quantum nonreciprocity, including implementation methods. We also introduce a new, very general metric for quantum nonreciprocity that uses quantum information-theoretic tools, and use this to characterize the nonreciprocity of our mechanism in the presence of imperfections. Our study also goes beyond just fundamental considerations. We discuss a potentially powerful application of our new directional dynamics: a method for realizing dissipative steady-state unitary quantum gate operations. By this, we mean here that the gate operation is realized in the dissipative steady state of the dynamics. The basic idea is sketched in Fig. 1(b). Starting from a bipartite system having control (A) and target (B) subsystems, the goal is for the dissipative relaxation of the system [under a dynamics of the form of Eq. (2)] to implement a unitary operation on B whose form is dictated by the initial state of A. Specifically, for a set of initial states for A indexed by λ, we can achieve

$$\begin{aligned}\hat{\rho}_B(\infty) &= \lim_{t\to\infty} \mathrm{Tr}_A[e^{t\mathcal{L}_{\rm dir}}(\hat{\rho}_A(\lambda)\otimes\hat{\rho}_B)] \\ &= \hat{U}_B(\lambda)\hat{\rho}_B\hat{U}_B(\lambda)^\dagger.\end{aligned} \tag{3}$$

The steady state of B is related to its initial state by a unitary, whose form is dictated by the initial A state. As we discuss in detail, this mechanism for performing gates has several attractive features: it does not require any timing control, and its dissipative nature makes it robust against certain kinds of errors in initial A state preparation (as long as all such A states lead to the same B unitary). We stress that the mechanism of Eq. (3) is completely distinct from previous works exploring alternative dissipative approaches to quantum control [24–31]. While the dissipative implementation of quantum gates in Eq. (3) is not robust against loss in A, intriguingly, recent work [32] shows that it leads to a promising route to realizing autonomous quantum error-correction schemes.

The remainder of the paper is organized as follows. In Sec. II, we set the stage for our discussions of quantum nonreciprocity by introducing a very general quantum information metric that quantifies quantum nonreciprocal (QNR) interactions in an arbitrary system, in a state-independent manner. Section III introduces our new mechanism for QNR in the simplest setting, and discusses its application to dissipative steady-state gate operations. Section IV discusses physical implementation strategies that are compatible with state-of-the-art superconducting circuit and quantum optical platforms. Section V generalizes our new mechanism to more complex cases with multiple dissipators, and demonstrates that these QNR interactions can generate entanglement. We conclude in Sec. VI.

II. QUANTIFYING NONRECIPROCITY OF GENERAL QUANTUM DYNAMICS

A. Basic notions

Before introducing our new mechanism, we start with a more basic question: what is a fundamental, system-agnostic way of identifying and quantifying nonreciprocal dynamics? While in simple linear settings one can just look at the asymmetry of scattering-matrix coefficients at

a particular input frequency, we would like a more general metric that can apply even when there is no obvious connection to scattering, and which is not contingent on a particular choice of initial state. As we now discuss, we can formulate such a metric using well-known quantum information-theoretic quantities.

We start with a generic bipartite system with subsystems A and B, whose dynamics are described by the evolution superoperator $\mathcal{E}_t^{(AB)}$. This superoperator tells us how the system density matrix evolves, i.e.,

$$\hat{\rho}_{AB}(t) = \mathcal{E}_t^{(AB)}\hat{\rho}_{AB}(0). \tag{4}$$

Here $\mathcal{E}_t^{(AB)}$ is also known as a quantum map or channel; any physical evolution corresponds to such a map, with the requirement that $\mathcal{E}_t^{(AB)}$ be completely positive and trace preserving [33]. We stress that this description encompasses a full range of dynamics from nondissipative unitary evolution to highly complex non-Markovian dissipative evolution.

We now want a metric that tells us whether the dynamics described by $\mathcal{E}_t^{(AB)}$ is nonreciprocal. We first introduce the *isolation function* of subsystem A, $I^{(A)}(t)$, which quantifies how sensitive the evolution of A is over this time interval to the initial state of subsystem B. The isolation function of subsystem B, $I^{(B)}(t)$, will be defined in an analogous manner. Isolation function $I^{(A)}(t)$ can be directly connected to a standard task in quantum information theory. Suppose that we first prepare subsystem B in one of two given states, $|\phi_1\rangle$ or $|\phi_2\rangle$ with equal probability. We also prepare A in some state $\hat{\rho}_A$. We then let the total system evolve for time t. We thus have two possible time-evolution maps for system A, contingent on the two subsystem B initial states:

$$\mathcal{E}^{(A)}_{|\phi_i\rangle}(t)\hat{\rho}_A \equiv \mathrm{Tr}_B[\mathcal{E}_t^{(AB)}(\hat{\rho}_A \otimes |\phi_i\rangle_B\langle\phi_i|)]. \tag{5}$$

The goal is now to optimally guess which initial state B we started with, using only a *single* measurement on the A system at time t. We are interested in the maximum success probability where we optimize over all A initial states as well as the final A subsystem measurement. This probability $p_{\max}(\{|\phi_1\rangle, |\phi_2\rangle\})$ gives us a measure of how different the A system dynamics is depending on the choice of initial B state. One finds that [34]

$$p_{\max}(\{|\phi_1\rangle, |\phi_2\rangle\}) = \tfrac{1}{2} + \tfrac{1}{4}\left\|\mathcal{E}^{(A)}_{|\phi_1\rangle}(t) - \mathcal{E}^{(A)}_{|\phi_2\rangle}(t)\right\|_\diamond. \tag{6}$$

Here, $\|\cdot\|_\diamond$ denotes the so-called diamond norm and provides a distance measure between two quantum channels [35], which is stable with respect to tensor product operations (i.e., attaching ancillary quantum systems to A). Note that $p_{\max}$ must lie in the interval $[0.5, 1]$.

Equation (6) thus provides a fundamental metric for the sensitivity of the A system dynamics to a change in the initial state of B. This then directly leads to a fundamental notion of how isolated the A system dynamics is from B: further optimize Eq. (6) over the choice of the B system initial states. This leads us to define the subsystem-A isolation as

$$I^{(A)}(t) \equiv 1 - \frac{1}{2}\max_{|\phi_1\rangle,|\phi_2\rangle\in\mathcal{H}_B}\left\|\mathcal{E}^{(A)}_{|\phi_1\rangle}(t) - \mathcal{E}^{(A)}_{|\phi_2\rangle}(t)\right\|_\diamond, \tag{7}$$

where $\mathcal{H}_B$ denotes the Hilbert space of B. Isolation function $I^{(A)}(t)$ lies in the interval $[0, 1]$, and measures the maximal influence a change in the initial subsystem-B state could have on the A subsystem dynamics. The case of complete isolation, $I^{(A)}(t) = 1$, implies that the dynamics of A is completely independent of the initial state of subsystem B. The isolation function for subsystem B is defined in a completely analogous manner. Note that if A and B are not coupled at all in the dynamics (i.e., the total channel is a tensor product of independent channels for each subsystem), then both subsystems are fully isolated at all times: $I^{(A)}(t) = I^{(B)}(t) = 1$. For $t = 0$, both systems are also of course always trivially isolated as the total channel is the identity.

B. Instantaneous nonreciprocity

These isolation functions now give us a simple way of identifying nonreciprocal dynamics as an evolution map that yields $I^{(A)}(t) \neq I^{(B)}(t)$, i.e., a situation where there is an asymmetry in how strongly A influences B versus how strongly B influences A. One can discuss nonreciprocity for the instantaneous quantum map at a specific time, as well as for the entire evolution. If we focus on a specific time, we can define $\mathcal{E}_t^{(AB)}$ as being reciprocal or nonreciprocal at time t using the isolation functions, i.e.,

$$\begin{aligned} & I^{(A)}(t) = I^{(B)}(t) \\ \Longrightarrow\ & \mathcal{E}_t^{(AB)} \text{ is instantaneously reciprocal at } t; \end{aligned} \tag{8}$$

$$\begin{aligned} & I^{(A)}(t) \neq I^{(B)}(t) \\ \Longrightarrow\ & \mathcal{E}_t^{(AB)} \text{ is instantaneously nonreciprocal at } t. \end{aligned} \tag{9}$$

C. Global nonreciprocity

One could also ask about whether the dynamics is nonreciprocal over an entire time interval $[0, t]$. In this case, we can define reciprocity by insisting that the isolations are identical over the entire time interval:

$$\begin{aligned} & I^{(A)}(t) = I^{(B)}(t) \quad \text{for all } t \in (0, +\infty) \\ \Longrightarrow\ & \text{dynamics is reciprocal;} \end{aligned} \tag{10}$$

$$\begin{aligned} & \exists t \in (0, +\infty) \text{ such that } I^{(A)}(t) \neq I^{(B)}(t) \\ \Longrightarrow\ & \text{dynamics is nonreciprocal.} \end{aligned} \tag{11}$$

D. Fully nonreciprocal dynamics

Finally, one is also often interested in identifying situations with full nonreciprocity, where one system is unaffected by the other, but is nonetheless still able to influence it. We first consider the quantum map at a specific time, and define instantaneous full nonreciprocity (i.e., unidirectionality) from A to B for $\mathcal{E}_t^{(AB)}$ at time t as

$$I^{(A)}(t) = 1, \qquad I^{(B)}(t) < 1 \tag{12}$$

$$\implies \mathcal{E}_t^{(AB)} \text{ is instantaneously unidirectional } (A \to B). \tag{13}$$

Physically, conditions (12) can be understood as ensuring that states of A can affect evolution of B, but not vice versa. One can also define maximal unidirectionality from A to B (at time t) as any evolution that yields $I^{(A)}(t) = 1$ and $I^{(B)}(t) = 0$. More generally, one can also define fully nonreciprocal dynamics via the condition that the dynamics of A is fully isolated at all times, but B is not fully isolated at some time, as

$$\forall t,\ I^{(A)}(t) = 1, \quad \text{and} \quad \exists t \text{ such that } I^{(B)}(t) < 1. \tag{14}$$

The case of B-to-A full nonreciprocity can be similarly defined by interchanging A and B in Eq. (14). We stress that having fully isolated A dynamics is a necessary but not sufficient condition for full nonreciprocity from A to B. In fact, it is possible to have dynamics generated by nontrivial interactions that is isolated in both directions, i.e., $I^{(A)}(t) = I^{(B)}(t) = 1$ (for an example, see Sec. III D).

E. Physical intuition and example cases

For a variety of simple test cases, our formal definitions of reciprocity and nonreciprocity agree with simple intuition. For example, it is easy to show that if the starting bipartite system is uncoupled, or is symmetric under permutation of A and B labels, then its dynamics is automatically reciprocal as per the definition in Eq. (10). Our definition also does more than simply quantify asymmetry of the bipartite system. As an example, in Appendix A 1 we consider a class of highly asymmetric bipartite, nondissipative systems that are always reciprocal as per our definition in Eq. (10). These systems take the B subsystem to be a qubit, the A system to be *arbitrary*, and take the two subsystems to interact via a coupling Hamiltonian that commutes with the B-only Hamiltonian. Another interesting test case is where A and B are both single qubits. In this case, if the evolution is an arbitrary unitary, then it must be fully reciprocal (see Appendix A 2).

To gain intuition about the opposite limit of full nonreciprocity, it is useful to examine cases where the dynamics of A is fully isolated. This is of course a necessary condition for fully unidirectional dynamics [see Eq. (14)], but is of course not sufficient. One can show that if B can be exactly traced out from the total system dynamics, then the A isolation by our definition stays unity throughout the time evolution, i.e.,

$$\begin{aligned} &\mathrm{Tr}_B[\mathcal{E}_t^{(AB)}(\hat{\rho}_A \otimes \hat{\rho}_B)] = \mathcal{E}_t^{(A)}\hat{\rho}_A \quad \text{for all } t \\ &\implies \quad I^{(A)}(t) \equiv 1 \quad \text{for all } t. \end{aligned} \tag{15}$$

Here $\mathcal{E}_t^{(A)}$ is a local superoperator acting on A and is independent of $\hat{\rho}_B$. As a result, the dynamics of a generic cascaded quantum system from A to B [see Eq. (1)] must be fully isolated in terms of subsystem A. Furthermore, because A cannot be exactly traced out from the system dynamics, the dynamics of B can be affected by A, so that dynamics generated by Eq. (1) is fully nonreciprocal by Eq. (14).

III. QUANTUM NONRECIPROCITY VIA GENERALIZED GAUGE SYMMETRY

A. Gauge-invariance nonreciprocity with a single dissipator

We now introduce our new method for realizing nonreciprocal quantum dynamics via an open-system Markovian dynamics that is *distinct* from cascaded quantum systems. We begin with the simplest case of a Lindblad master equation with a single dissipator, leaving generalizations to Sec. V. We start with a seemingly trivial observation for a single, generic Lindblad dissipator on system A. Such a dynamics is described by

$$\mathcal{L}_{A,1}\hat{\rho}_A = \Gamma\mathcal{D}[\hat{A}]\hat{\rho}_A. \tag{16}$$

It is straightforward to see that this Lindbladian is invariant under an arbitrary gauge transformation of the jump operator $\hat{A} \to \hat{A}e^{i\theta(t)}$, where $\theta(t)$ can be an arbitrary time-dependent real function. This invariance formally corresponds to a local (in time) gauge symmetry of a generic Lindblad dissipator.

We can use this trivial insensitivity of the dynamics to $\theta(t)$ to now obtain a nonreciprocal interaction between two systems: simply replace the classical time-dependent phase with a quantum operator acting on a different quantum system B: $\theta(t) \to \hat{\theta}_B$. As shown in Fig. 1(a), we now rewrite the phase factor in the jump operator as unitary operator $\hat{U}_B$ acting on subsystem B. We thus obtain a new QME [see also Eq. (2)]:

$$\mathcal{L}_{\mathrm{dir}}\hat{\rho} = \Gamma\mathcal{D}[\hat{A}\hat{U}_B]\hat{\rho}. \tag{17}$$

One can easily show that the gauge invariance property discussed above ensures that the dynamics of A is insensitive to B. More explicitly, consider a general master equation where the interaction between A and B is given by Eq. (17): $d\hat{\rho}/dt = (\mathcal{L}_{A,i} + \mathcal{L}_{B,i})\hat{\rho} + \Gamma\mathcal{D}[\hat{A}\hat{U}_B]\hat{\rho}$, with

$\mathcal{L}_{A(B),i}$ describing the internal dynamics of A (B). One can exactly trace out B to obtain a QME for the A reduced density matrix $\hat{\rho}_A = \mathrm{Tr}_B\hat{\rho}$ alone, as

$$\frac{d\hat{\rho}_A}{dt} = \mathcal{L}_{A,i}\hat{\rho}_A + \Gamma\mathcal{D}[\hat{A}]\hat{\rho}_A. \tag{18}$$

However, the converse is in general not true: B will in general be influenced by A, i.e., its evolution is sensitive to the initial state of A as well as $\mathcal{L}_{A,i}$. The only exception is the case where $\hat{A}$ is proportional to a unitary; see also Sec. III D.

The more formal definitions of nonreciprocity introduced in Sec. II also yield an identical picture. Because the equation of motion of the A subsystem is independent of the B state [see Eq. (18)], it follows that the A isolation must be unity throughout time evolution, i.e., $I^{(A)}(t) \equiv 1$ for all t. For the B isolation, assuming that $\hat{A}$ is not proportional to a unitary operator, one can generally show that $I^{(B)}(t) < 1$ for some time t; see also Sec. III B for a concrete example with a bosonic lowering operator as $\hat{A}$. Thus, according to the new metric based on isolation functions, the QME in Eq. (17) describes fully nonreciprocal dynamics from A to B as long as we have $\hat{A}^\dagger\hat{A} \not\propto \hat{\mathbb{I}}$.

We stress that Eq. (17) describes a generic nonreciprocal open-system dynamics that is distinct from a cascaded quantum system: it *cannot* be written in the form of a cascaded QME, Eq. (1). Our new approach in Eq. (17) can be written as a Liouvillian that has no Hamiltonian part, and that has a single dissipator with a jump operator that is a *product* of an A operator and a B operator. In marked contrast, the cascaded quantum systems QME in Eq. (1) has a Hamiltonian in its Liouvillian, and a jump operator that is the *sum* of a subsystem-A operator and a subsystem-B operator. These cannot be made equivalent. At a more physical level, the differences in jump operators correspond to different forms of system-bath coupling. The inequivalence also implies that the nonreciprocal interaction described by Eq. (17) *cannot* be realized by coupling A and B to a directional waveguide [see Fig. 1(a)].

B. Example: photon-loss dissipator

To make our ideas more concrete, consider a simple case where the A subsystem in Eq. (17) is a bosonic mode, and $\hat{A}$ is taken to be the photon lowering operator $\hat{a}$ for this mode. Furthermore, take an initial state where A is unentangled with B, and is prepared either in the vacuum state $|0\rangle$, or in the Fock state $|\ell\rangle$ ($\ell > 0$). From Eq. (7), we can thus obtain an upper limit of the corresponding B isolation in the long-time $t \to \infty$ limit as

$$I^{(B)}(\infty) \leq 1 - \frac{1}{2}\lim_{t\to\infty}\left\|\mathcal{E}^{(B)}_{|0\rangle}(t) - \mathcal{E}^{(B)}_{|\ell\rangle}(t)\right\|_\diamond. \tag{19}$$

One can also show (see Sec. III C) that the subsystem-B evolution maps appearing in this equation have an extremely simple form:

$$\lim_{t\to\infty}\mathcal{E}^{(B)}_{|n\rangle}(t)\hat{\rho}_B = \hat{U}_B^n\hat{\rho}_B\hat{U}_B^{\dagger n}. \tag{20}$$

Intuitively, this describes a dissipative process where each time a photon is lost from the A cavity, subsystem B undergoes a unitary evolution $\hat{U}_B$. We can thus derive an upper bound for the corresponding B isolation in the long-time $t \to \infty$ limit as [36]

$$I^{(B)}(\infty) \leq 1 - \sqrt{1 - \min_{|\phi\rangle\in\mathcal{H}_B}|\langle\phi|\hat{U}_B^\ell|\phi\rangle|^2}. \tag{21}$$

Letting $e^{i\beta_m}$ denote the eigenvalues $\hat{U}_B$, the right-hand side of Eq. (21) can be further rewritten explicitly as

$$I^{(B)}(\infty) \leq 1 - \max_{m,n}\left|\sin\frac{\ell(\beta_m - \beta_n)}{2}\right|. \tag{22}$$

Thus, the B isolation in the long-time limit is less than 1 for any nontrivial unitary $\hat{U}_B^\ell$ (i.e., not proportional to the identity map), signaling nontrivial influence from A to B. The isolation reaches a minimal value of zero if $\hat{U}_B^\ell$ has two eigenvalues with relative π phase difference, in which case the long-time evolution becomes maximally nonreciprocal.

C. Dissipative quantum gates mediated by the new form of nonreciprocal interaction

Our new nonreciprocal QME has many interesting features. Here, we focus on a potentially powerful application: the implementation of unitary gate operations on subsystem B that are realized in a dissipative steady state, and whose form is controlled by the initial state of subsystem A. The most generic way to realize this is to construct dynamics of the form (17), where $\hat{A}$ has a subspace of dark states $\mathcal{D}$: if $|d\rangle \in \mathcal{D}$, then $\hat{A}|d\rangle = 0$. Furthermore, let $\mathcal{S}$ denote the set of states in the intersection between the orthogonal complement of $\mathcal{D}$ and the inverse image of $\mathcal{D}$ under $\hat{A}$. A given state $|\psi\rangle \in \mathcal{S}$ is both orthogonal to the dark state subspace and has the property that $\hat{A}|\psi\rangle$ is in $\mathcal{D}$ (i.e., a single action of $\hat{A}$ results in a dark state). We now have a simple way to obtain our dissipative gate.

(a) At $t = 0$ the full system is taken to be in a product state $\hat{\rho}_{AB}(0) = \hat{\rho}_A(0) \otimes \hat{\rho}_B(0)$.
(b) If we do not want a gate operation to be performed on B, we start subsystem A in an arbitrary dark state in $\mathcal{D}$. In this case, there is no evolution under Eq. (17), and the subsystem-B state is unchanged.
(c) To turn the gate on, we instead prepare subsystem A in an arbitrary state in $\mathcal{S}$. In this case, there is nontrivial evolution under Eq. (17). To achieve the gate

operation, one just waits until the system reaches its steady state. The dissipative steady state will be

$$\hat{\rho}_{AB}(\infty) = \hat{\rho}'_A \otimes (\hat{U}_B \hat{\rho}_B(0) \hat{U}_B^\dagger), \tag{23}$$

where $\hat{\rho}'_A$ is in $\mathcal{D}$, i.e., it is a dark state. The final state of B is related to the initial state by the unitary $\hat{U}_B$.

We stress that this approach realizes a gate operation on system B in the dissipative steady state; no precise timing control is needed. The only control that is needed is to prepare subsystem A at $t = 0$ in a state in subspace $\mathcal{S}$. Crucially, this control need not be perfect, as any state in this manifold (pure or impure) will lead to the desired gate operation. This resilience is similar in spirit to recent theoretical ideas for mixed-state encodings of quantum information (see, e.g., Ref. [37]).

An even more versatile kind of controllable dissipative gate is possible if $\hat{A}$ has the general structure of a lowering operator. By this, we mean that, within a given subspace, $\hat{A}$ is a matrix that only has nonzero entries along the superdiagonal. This is exactly the situation we have if $\hat{A}$ is a bosonic lowering operator $\hat{a}$; hence, we consider this case in what follows. As illustrated in Fig. 1(b), if the control subsystem A is initialized in a Fock state $|\ell\rangle$, the long-time dynamics of the target system effectively applies $\hat{U}_B$ ℓ times on the initial target state, so that we have

$$\lim_{t\to\infty} \mathrm{Tr}_A\{e^{\mathcal{L}_{\mathrm{dir}} t}[|\ell\rangle\langle\ell| \otimes \hat{\rho}_B(0)]\} = (\hat{U}_B)^\ell \hat{\rho}_B(0) (\hat{U}_B^\dagger)^\ell. \tag{24}$$

This recipe allows one to apply, e.g., tunable phase gates on a target qubit. We stress that this gate mechanism works for generic systems (see Appendix G for an example where it can be used to apply displacement gates on a harmonic oscillator).

We note that this approach to dissipative unitary gate operations is distinct from previous works exploring dissipative quantum control. For example, Verstraete *et al.* [24] focused on dissipatively realizing a *unique* steady state that effectively realizes a quantum computational task. In contrast, we are dissipatively implementing a steady-state unitary operation on subsystem B, not a unique state. Our dissipator in Eq. (17) has of course multiple steady states, something that is exploited by our protocol. Approaches for mimicking Hamiltonian evolution on a subsystem have also been formulated, using either strong dissipation [25–27], measurements [28–30], or fast repetitive resets [31]. In stark contrast to our work, these approaches do not yield a time-independent steady state. Instead, one needs to explicitly shut off the dynamics at a particular time in order to achieve a particular unitary (whereas we achieve the unitary in the long-time steady state). We stress that our dissipative unitary gate does not give rise to long-time oscillations; as such it is completely distinct from recent mechanisms studied in the context of quantum synchronization [38,39].

The dissipative steady-state gate mechanism described here suggests a fundamentally different physical architecture for constructing a quantum processor. In the standard architectures, the analog information parameterizing a target gate operation $\hat{U}_B$ (e.g., the angles and axes of a single-qubit rotation) is contained in external control pulses, whose precision is subject to fluctuations of external control electronics. Here, the analog parameters of $\hat{U}_B$ are "hardwired" within the engineered quantum system, whose precision is an inherent property of the quantum device itself. Subsystem A acts as a classical switch for the gate, and the only information flow needed to execute the quantum gate is a binary command. While it may be nontrivial to prepare subsystem A in the starting states in the basic examples presented in this paper (Fock states), in principle, the starting manifold $\mathcal{S}$ can be made large and macroscopically distinguishable from $\mathcal{D}$, so the control need not be perfect as any state in this $\mathcal{S}$ (pure or impure) will lead to the desired gate operation. Finally, to be clear, the nonreciprocal dissipator in the current scheme does not counter decoherence in $\hat{\rho}_B$. Interestingly, the mechanism of steady-state unitary gates here provides a powerful knob for implementing autonomous quantum error correction [40–44] using reservoir engineering; see Ref. [32].

D. Gauge-invariance nonreciprocity: other generic cases

We now discuss the physics of our nonreciprocal dynamics in Eq. (17) for different generic choices of the subsystem-A operator (beyond the lowering operator case discussed above). Consider first the case where $\hat{A}$ is itself unitary. In this case, there is no asymmetry in our dissipator (i.e., both subsystem operators are unitary), and correspondingly we would expect that there cannot be any directional interaction. This is indeed what occurs: in this case, it is easy to confirm that both systems are isolated from one another. The only way to see signatures of the interaction would be to consider the evolution of correlations between them. As discussed in Refs. [45,46], this type of dissipative dynamics can be understood as coupling both A and B to the same classical Poisson point process. It is then straightforward to show that the time evolution of both subsystems is independent of the other, and the correlation between classical stochastic processes coupled to A versus B is only discernible if one looks at AB correlators.

Another general case is where the $\hat{A}$ operator in Eq. (17) is Hermitian. While the dynamics in this case is directional from A to B, we can exactly solve for the time evolution in terms of the eigenbasis of the jump operator. The system dynamics now allows a simple interpretation, i.e., B

dephases at rates depending on the A state, but not vice versa.

E. Connection to measurement-and-feedforward processes

Given that a large class of standard quantum cascaded systems can be intuitively understood as being equivalent to a measurement-and-feedforward (MF) process [23,47], it is worth discussing the relation between our nonlinear dissipator in Eq. (17) and MF protocols. As shown in Ref. [47], dissipators given by Eq. (17) can be realized via a generalized MF process as $e^{\mathcal{L}_{\rm dir}\delta t}\hat{\rho} = \sum_{\ell=1,2}\hat{M}_\ell\hat{\rho}\hat{M}_\ell^\dagger$. Here, the Kraus operators $\hat{M}_\ell$ are given by (to order δt)

$$\hat{M}_1 = \sqrt{\Gamma\delta t}\hat{A}\hat{U}_B, \qquad \hat{M}_2 = 1 - \Gamma\hat{A}^\dagger\hat{A}\delta t, \tag{25}$$

which satisfy the normalization condition $\sum_{\ell=1,2}\hat{M}_\ell^\dagger\hat{M}_\ell = \hat{\mathbb{I}}$. Intuitively, this stochastic process corresponds to weakly measuring A, and subsequently applying unitary transformations on B conditioned on the measurement results. This interpretation provides a simple, complementary way to understand the directionality of our dynamics in the single-dissipator case. It also tells us that this dynamics can never generate entanglement between A and B.

While, for a single dissipator, both the gauge-invariance picture and MF picture let us understand the directionality, the same is not true for the multiple-dissipator case analyzed in Sec. V. In this case, the gauge invariance pictures ensures nonreciprocity, but there is no mapping onto a MF process (and, in fact, the dynamics can create entanglement).

F. Non-Markovian effects

As we have stressed, the nonreciprocity of the dissipative dynamics in our basic dissipator of Eq. (17) is directly related to the local gauge invariance of a standard Lindblad dissipator. This effective gauge symmetry however only emerges in the limit of a Markovian bath, something we discuss in detail in Appendix B. Physically, it requires the bath correlation time τ_E to be much smaller than the timescale associated with variation of the gauge phase $\theta(t)$: $\tau_E \ll [\dot{\theta}(t)]^{-1}$. If this condition is met, then the bath is effectively only sensitive to the instantaneous value of $\theta(t)$, and there is no difference between a constant in time $\theta(t)$ versus a time-varying phase. Conversely, if changes in the gauge phase $\theta(t)$ are not negligible during the bath correlation time, the dynamics of $\theta(t)$ will induce non-Markovian effects in the bath, and the system dynamics will no longer be gauge invariant.

The above picture can be made rigorous, and one can calculate leading non-Markovian corrections. In Appendix C 1, we consider a microscopic bath model with a system-environment (SE) interaction Hamiltonian of the form $\hat{H}_{\rm SE} = e^{j\theta(t)}\hat{\xi}^\dagger\hat{A} + \text{H.c.}$ Here, $\hat{\xi}$ is the bath operator that couples to the system. We derive the leading-order correction to Eq. (16) due to a finite bath correlation time τ_E, a correction that scales as $\tau_E\dot{\theta}(t)$. Furthermore, in Sec. IV B, we work with an explicit quantum realization of our dissipative scheme, and use the isolation functions defined in Sec. II to quantify how a finite bath correlation time causes deviations from full nonreciprocity.

IV. PHYSICAL IMPLEMENTATION IN CAVITY QED SYSTEMS

A. Basic setup

We now discuss methods for implementing the general nonreciprocal dynamics of Eq. (17) in a quantum optical setup. Note that, incidentally, dissipators of the form (17) have been used to describe, e.g., decay of two-level atoms with recoil [48,49]. However, to the best of our knowledge, such processes have not been discussed in the context of engineering useful nonreciprocal interactions. The standard treatment of such processes also immediately expands out the unitary in Eq. (17) (see, e.g., Ref. [50]), obscuring the fully nonreciprocal structure. In this section, in contrast, we focus on experimentally compatible methods for realizing a generic target dissipator. One direct approach would be to explicitly break TRS, and use standard nonreciprocal elements like a circulator or a chiral waveguide. Formally, such implementations involve starting with a larger system described by a cascaded master equation, eliminating degrees of freedom, and then obtaining the effective dynamics of Eq. (17). We discuss a generic method for doing this (starting with a chiral waveguide) in Appendix D. Note that several recent circuit QED experiments using nonreciprocal elements could be usefully interpreted in this way [51–53] (see Appendix D for more details). We also stress that this recipe for realizing Eq. (17) using a larger cascaded quantum system does not imply that the two types of dynamics are equivalent. To see this, one may consider an analogical example involving open versus closed quantum systems: there, it is straightforward to understand that the two still correspond to very distinct forms of dynamics, even though any open quantum system can be realized by starting from a larger closed quantum system with auxiliary reservoir modes, and then eliminating the reservoir degrees of freedom.

We focus here on a more intriguing implementation strategy that uses reservoir engineering techniques but *does not require any elements that explicitly break time-reversal symmetry, i.e., the dynamics does not involve any nontrivial gauge-invariant phases*. We take subsystem A to be a resonator mode with bosonic annihilation operator $\hat{a}$, and subsystem B to be a qubit with Pauli-z operator $\hat{\sigma}_z$. We wish to realize our nonreciprocal master equation (17)

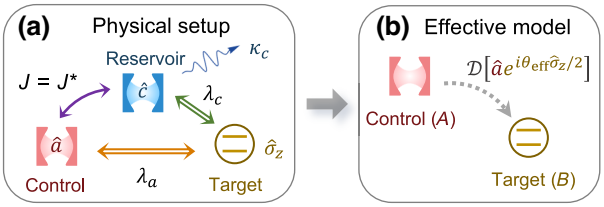

FIG. 2. (a) Schematic of a setup that realizes a nonreciprocal interaction from a cavity mode a (A) to a qubit (B). Both subsystems are coupled to an auxiliary damped bosonic mode c that plays the role of a reservoir; see Eq. (28). (b) The effective dissipator describing the nonreciprocal interaction in the Markovian reservoir limit $\kappa_c \gg J$.

with the choices $\hat{A} = \hat{a}$ and $\hat{U}_B = \exp(-i\theta\hat{\sigma}_z/2)$, i.e.,

$$\frac{d\hat{\rho}}{dt} = \mathcal{L}_{\rm gate}\hat{\rho} = \Gamma\mathcal{D}[e^{-i\theta\hat{\sigma}_z/2}\hat{a}]\hat{\rho}. \tag{26}$$

To engineer the effective dynamics in Eq. (26), we couple both the cavity mode a and the qubit to an auxiliary, highly damped bosonic mode $\hat{c}$ (decay rate κ_c), via tunneling and dispersive interactions, respectively [see Fig. 2(a)]. The interaction Hamiltonian between the system and the c mode (i.e., the reservoir) is

$$\hat{H}_{\rm int} = (J\hat{a}^\dagger\hat{c} + \text{H.c.}) + (\lambda_c/2)\hat{\sigma}_z\hat{c}^\dagger\hat{c}, \tag{27}$$

where J denotes the complex tunnel coupling rate. We also include a direct Hamiltonian dispersive coupling between the cavity mode $\hat{a}$ and the qubit, $\hat{H}_S = (\lambda_a/2)\hat{\sigma}_z\hat{a}^\dagger\hat{a}$. We take the a and c modes to be resonant, and work in a rotating frame where the a, c and qubit frequencies are shifted to zero. The total dynamics (including the reservoir c mode) is then described by the QME

$$\frac{d\hat{\rho}_{\rm tot}}{dt} = \mathcal{L}_{\rm SR}\hat{\rho}_{\rm tot} = -i[\hat{H}_S + \hat{H}_{\rm int}, \hat{\rho}_{\rm tot}] + \kappa_c\mathcal{D}[\hat{c}]\hat{\rho}_{\rm tot}. \tag{28}$$

Note that Eq. (28) does not involve any explicit breaking of TRS, in that there is no nontrivial gauge phase. Even if J is complex, the corresponding hopping phase can always be eliminated by a gauge transformation on $\hat{a}$; hence, the phase of J plays no role. This can be understood physically from the fact that the setup in Eq. (28) does not host any closed loops enclosing a nontrivial flux. We thus assume henceforth, without loss of generality, that the cavity-reservoir coupling amplitude in Eq. (28) is real and positive, i.e., $J = |J|$.

We next consider the limit where reservoir-mode photons decay much faster than their tunneling rate to the cavity, i.e., $\kappa_c \gg J$. We can then adiabatically eliminate the reservoir (see Appendix E), yielding an effective QME for the cavity-qubit density matrix $\hat{\rho}$:

$$\frac{d\hat{\rho}}{dt} = -i\left[\frac{\lambda_a + \lambda_{\rm eff}}{2}\hat{\sigma}_z\hat{a}^\dagger\hat{a}, \hat{\rho}\right] + \Gamma_{\rm eff}\mathcal{D}[e^{-i\theta_{\rm eff}\hat{\sigma}_z/2}\hat{a}]\hat{\rho}. \tag{29}$$

For $\kappa_c \gg J$, the parameters in this QME are

$$\Gamma_{\rm eff} = 4J^2\kappa_c/(\kappa_c^2 + \lambda_c^2), \tag{30}$$

$$\lambda_{\rm eff} = -4J^2\lambda_c/(\kappa_c^2 + \lambda_c^2), \tag{31}$$

$$\theta_{\rm eff} = 2\arctan(\lambda_c/\kappa_c). \tag{32}$$

The various couplings in Eq. (29) can easily be given physical interpretations. In the regime $\kappa_c \gg J$, the reservoir and the qubit together form a new effective, Markovian environment for the cavity mode. The corresponding effective cavity decay rate $\Gamma_{\rm eff}$ matches Fermi's golden rule expectation, and is independent of the qubit state. Here $\lambda_{\rm eff}$ is an induced dispersive coupling arising from weak hybridization of a and c modes. The most interesting parameter is the phase $\theta_{\rm eff}$. At a heuristic level, whenever a photon hops from the cavity mode to the reservoir mode and subsequently decays, the qubit is rotated by an angle $\theta_{\rm eff}$ about the z axis. In the limit $\lambda_c \ll \kappa_c$, this phase shift can be understood as a product between photon dwell time in the reservoir c mode, $\tau_c \sim \kappa_c^{-1}$, and the bare qubit-reservoir dispersive coupling strength λ_c.

Finally, we imagine tuning the direct dispersive interaction so that it cancels the induced dispersive interaction, i.e., tune $\lambda_a = -\lambda_{\rm eff}$. In this case, we are left only with the dissipator in Eq. (29), which corresponds exactly to the form in our general directional QME (26). We thus obtain a completely directional dynamics from the cavity to the qubit. We stress that this physical implementation uses standard forms of qubit-cavity coupling, and does not use any explicitly nonreciprocal elements.

It is worth stepping back to ask what the essential ingredients are here to obtain quantum nonreciprocity without any explicit breaking of TRS. Like in our general QME in Eq. (2), it is crucial to have a final dissipator that is a product of A and B operators, corresponding to a nonlinear system-bath interaction (i.e., the environment couples to the composite operator $\hat{A}\hat{U}_B$). Furthermore, we need asymmetry: the B operator is unitary, the A operator is not. It is interesting to note that nonlinearity and broken inversion symmetry have been used in a very different manner to engineer nonreciprocal scattering without breaking TRS, both in classical [54,55] and quantum [56–58] settings. However, the scattering in those works are only approximately nonreciprocal, and even then only for a limited range of incident field powers and frequencies. This is very different from our mechanism, which is unidirectional independent of the initial state, and which is not equivalent to a simple scattering problem.

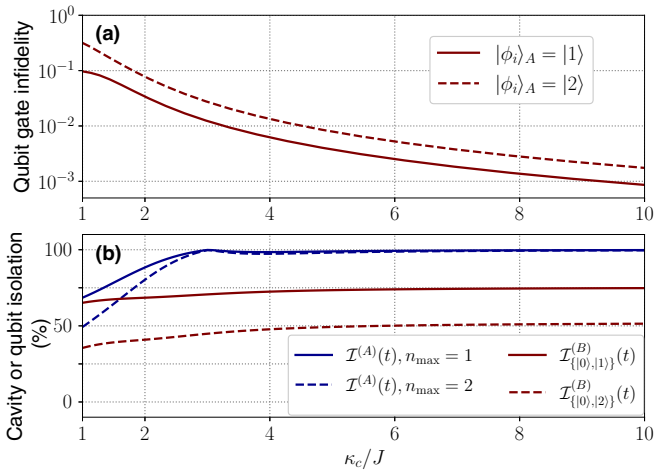

FIG. 3. (a) State-averaged infidelity of the cavity-controlled steady-state qubit gate, as a function of the non-Markovianity parameter κ_c/J. We consider different choices of initial cavity state: either the Fock state $|\phi_i\rangle_A = |1\rangle$ (solid) or $|2\rangle$ (dashed), and take $\theta_{\text{eff}} = \pi/6$. These different initial cavity states result in different gate operations. Infidelity tends to zero in the Markovian limit. (b) Isolation functions for both qubit and cavity as a function of the non-Markovianity parameter κ_c/J. We take a fixed evolution time $t = \pi/\Gamma_{\text{eff}}$ and $\theta_{\text{eff}} = \pi/6$. Blue curves denote the cavity isolation $I^{(A)}(t)$; cf. Eqs. (7) and (33). Red curves denote the conditional qubit isolation function $I^{(B)}_{\{|0\rangle,|\ell\rangle\}}(t)$ ($\ell = 1, 2$) that is an upper bound for the qubit isolation $I^{(B)}(t)$; see Eq. (34). We see strongly nonreciprocal behavior [i.e., $I^{(A)}(t) \simeq 1, I^{(B)}(t) < 1$] even away from the Markovian limit. As discussed in the main text, when calculating $I^{(A)}(t)$, we truncate the cavity Fock space to $n_{\max}$ photons for simplicity; the solid (dashed) curve is for $n_{\max} = 1$ ($n_{\max} = 2$).

B. Non-Markovian effects in the qubit-cavity setup

The physical implementation given by Eq. (28) provides a concrete setup where we can quantitatively analyze the effects of non-Markovianity. In this case, the reservoir (i.e., the highly damped c mode) has a finite correlation time $\tau_c \sim \kappa_c^{-1}$, and the Markovian limit is only reached for a large decay rate κ_c. As discussed in Sec. III F, the gauge symmetry that leads to isolation and nonreciprocity in our system only exists in the Markovian limit. We thus expect to see deviations from ideal behavior away from this limit.

In Fig. 3(a), we numerically investigate the infidelity [59] of the steady-state gate operation performed on B, as a function of the non-Markovianity parameter κ_c/J. The remaining parameters are also varied to keep the qubit gate angle θ in Eq. (32) fixed (here at a value $\theta_{\text{eff}} = \pi/6$). As expected, the infidelity rapidly drops to zero in the Markovian large-κ_c limit. This general trend remains true no matter what the chosen value of θ_{eff}.

We can also ask how non-Markovian effects impact nonreciprocity in this system. The isolation functions introduced in Sec. II let us quantitatively compare the nonreciprocity of Eq. (28) for different values of κ_c. For cavity dynamics due to Eq. (28), one can show that the cavity isolation function can be expressed as

$$I^{(A)}(t) = 1 - \tfrac{1}{2}\left\|\mathcal{E}^{(A)}_{|\uparrow\rangle}(t) - \mathcal{E}^{(A)}_{|\downarrow\rangle}(t)\right\|_\diamond, \tag{33}$$

where $|\uparrow\rangle, |\downarrow\rangle$ are $\hat{\sigma}_z$ eigenstates. While the diamond norm can be calculated for quantum maps acting on an infinite-dimensional Hilbert space, to make the problem numerically tractable, we truncate the cavity Hilbert space to have at most one or two photons (as this is already sufficient to illustrate the effect of non-Markovianity). The numerically calculated cavity isolations for increasing cavity decay rates κ_c are plotted with blue curves in Fig. 3(b). We see that, even for modest values of κ_c (i.e., not strongly in the Markovian regime), the cavity is well isolated.

To characterize nonreciprocity, we also need to consider the qubit isolation $I^{(B)}(t)$. We can find a simple upper bound for this quantity using the conditional qubit isolation $I^{(B)}_{\{|0\rangle,|\ell\rangle\}}(t)$, corresponding to initial control cavity Fock states $\{|0\rangle, |\ell\rangle\}$ ($\ell = 1, 2$), as

$$I^{(B)}(t) \leq \left(I^{(B)}_{\{|0\rangle,|\ell\rangle\}}(t) \equiv 1 - \tfrac{1}{2}\left\|\mathcal{E}^{(B)}_{|0\rangle}(t) - \mathcal{E}^{(B)}_{|\ell\rangle}(t)\right\|_\diamond\right). \tag{34}$$

From the definition of qubit isolation [cf. Eq. (7)], one sees that $I^{(B)}_{\{|0\rangle,|\ell\rangle\}}(t)$ must be no less than the actual qubit isolation function $I^{(B)}(t)$. As shown in Fig. 3(b), in the fast reservoir limit $\kappa_c/J \gg 1$, $I^{(B)}_{\{|0\rangle,|\ell\rangle\}}(t)$ ($\ell = 1, 2$) (red curves) is considerably smaller than cavity isolation $I^{(A)}(t)$ (blue curves), demonstrating that the reservoir mode mediates an effective unidirectional interaction from the control cavity to qubit.

V. GENERALIZED GAUGE-INVARIANCE NONRECIPROCITY: THE NON-ABELIAN CASE

A. Gauge symmetry of a multidissipator Lindblad master equation

We now discuss how our recipe for quantum nonreciprocity based on gauge invariance can be extended from Eq. (17) to a much broader class of dynamics. This generalized version involves dissipative dynamics with multiple dissipators, and the relevant local gauge symmetry can become non-Abelian. As we show, this generalized version is in general *not* equivalent to unconditional evolution under measurement and feedforward, and is capable of generating entanglement. It is also (like the single-dissipator case) distinct from the cascaded quantum systems master equation.

Similar to Sec. III A, we start by considering a single system A undergoing dissipative Lindblad dynamics, but

now involving multiple dissipators:

$$\frac{d}{dt}\hat{\rho}_A = \Gamma \sum_{\ell=1}^{N} \mathcal{D}[\hat{A}_\ell]\hat{\rho}_A \equiv \mathcal{L}_A \hat{\rho}_A. \tag{35}$$

As is well known, multidissipator Liouvillians like $\mathcal{L}_A$ are invariant under a wide class of transformations that mix the jump operators $\hat{A}_l$. Let u_{lm} be the matrix elements of an arbitrary $N \times N$ complex unitary matrix $\breve{\mathcal{U}}$ [60]. Then we necessarily have (see, e.g., Ref. [61])

$$\sum_{\ell=1}^{N} \mathcal{D}[\hat{A}_\ell] = \sum_{\ell=1}^{N} \mathcal{D}\Big[\sum_{m=1}^{N} u_{\ell m}\hat{A}_m\Big]. \tag{36}$$

The above invariance of $\mathcal{L}_A$ also trivially continues to hold if we make the unitary mixing matrix $\breve{\mathcal{U}}(t)$ time dependent. Formally, this represents a local-in-time, non-Abelian gauge symmetry, as, for $N \geq 1$, different allowed $\breve{\mathcal{U}}$ do not commute with one another.

For what follows, it will be helpful to give an intuitive picture of this multidissipator gauge symmetry. Each of the N dissipators in our master equation (indexed by ℓ) can be interpreted as describing the influence of an independent dissipative bath. The operator $\sum_m u_{\ell m}\hat{A}_m$ is then interpreted as the particular system-A operator that couples to bath ℓ. The $u_{\ell m}$ are thus system-bath coupling constants. Equation (36) hence tells us that there are several distinct ways to couple our system to the N baths that results in identical dynamics; all that is required is that the $u_{\ell m}$ form a unitary matrix. At a physical level, this means that each independent bath should couple to an "orthogonal" set of system operators (i.e., the rows of $\breve{\mathcal{U}}$ are orthogonal), and that the "total coupling strength" to bath l is always the same, i.e., $\sum_m |u_{lm}|^2 = 1$.

B. From gauge symmetry to a multidissipator nonreciprocal interaction

Similar to Sec. III A, Eq. (36) now provides a route to construct a nonreciprocal interaction with a second subsystem B: we make each matrix element $u_{\ell m}(t)$ of $\breve{\mathcal{U}}(t)$ an operator $\hat{u}_{\ell m}$ acting on B. This results in a new master equation acting on the state of the bipartite A plus B system:

$$\frac{d}{dt}\hat{\rho}_{AB} = \Gamma \sum_{\ell=1}^{N} \mathcal{D}[\hat{z}_\ell]\hat{\rho} \equiv \mathcal{L}_{\text{multi}}\hat{\rho} \tag{37}$$

with

$$\hat{z}_\ell = \sum_{m=1}^{N} \hat{A}_m \hat{u}_{\ell m}. \tag{38}$$

At a physical level, we can interpret this as a kind of generalized dissipative parametric coupling: system B controls the strength and form of the coupling between system A and the N dissipative baths in the problem. If the different $\hat{u}_{\ell m}$ operators fail to commute, then there will be unavoidable quantum noise in the magnitude and form of these couplings. Note that simpler parametric dissipative couplings have been studied in quantum optomechanics (see, e.g., Refs. [62,63]), where a mechanical resonator controls the loss rate of a photonic cavity mode.

The dissipative parametric coupling between A and B in Eqs. (37) and (38) will not in general be directional; this requires a further constraint. We now come to the central result of this section: the master equation (37) mediates a fully nonreciprocal interaction from A to B if the B operators satisfy the generalized unitarity constraint

$$\sum_{\ell=1}^{N} (\hat{u}_{\ell m})^\dagger \hat{u}_{\ell m'} = E_B \delta_{mm'} \hat{\mathbb{I}}_B, \tag{39}$$

where $\hat{\mathbb{I}}_B$ is the identity operator on subsystem B and E_B is a positive real constant. Noting that E_B can be absorbed into the definition of $\hat{u}_{\ell m}$, we set $E_B = 1$ going forward. If Eq. (39) is satisfied, it is easy to show that A is completely isolated from B: one can trace out B and derive a closed QME for the dynamics of A that is independent of any additional local dynamics acting on B. This isolation reflects the underlying gauge symmetry discussed above (see Appendix F 1 for details). The converse is not true: B will in general be influenced by A. Note that if Eq. (39) is satisfied, operators $\hat{u}_{\ell m}$ can be viewed as matrix elements of a generalized unitary transformation acting on a larger space (see Appendix F 1).

We stress that Eq. (37) is not a trivial generalization of the single-dissipator case in Eq. (17), since each individual dissipator $\mathcal{D}\left[\hat{z}_\ell\right]$ need not generate fully nonreciprocal dynamics on its own. Unidirectionality is thus in general a collective property of the full Liouvillian, and not of each dissipator on its own. Furthermore, there is no clever transformation that allows one in general to express the Liouvillian in such a form, i.e., $\mathcal{L}_{\text{multi}}\hat{\rho} \neq \sum_\ell \Gamma_\ell \mathcal{D}[\hat{A}'_\ell \hat{U}_{B,\ell}]\hat{\rho}$. At a physical level, Eq. (39) represents a constraint on our dissipative parametric interaction. While system B still parametrically controls the coupling of system A to the N dissipative baths in our problem, the coupling to each of these baths is fixed in magnitude irrespective of the state of B. Furthermore, each bath always sees an orthogonal combination of the system-A operators $\hat{A}_\ell$. This constraint ensures that system A is independent of system B [by virtue of the symmetry in Eq. (36)], but still allows A to influence B.

C. Example: qubit-controlled photonic loss

As a concrete example, we consider a bipartite system where the A subsystem is comprised of two photonic cavity modes a_1, a_2, and the B subsystem is a single qubit

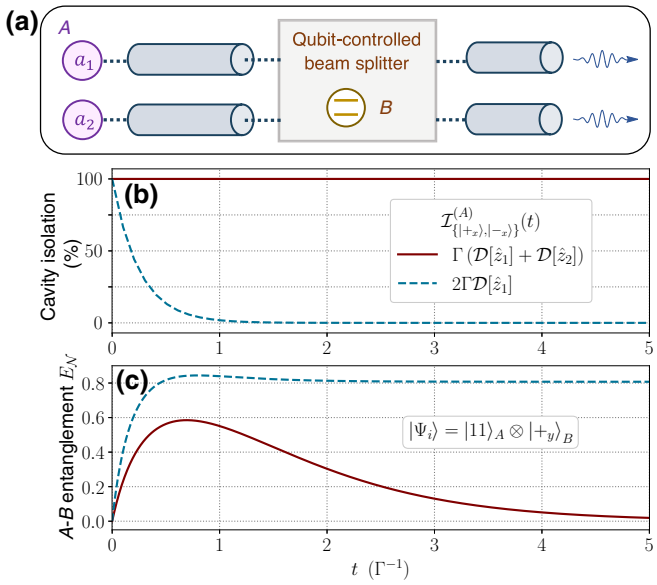

FIG. 4. (a) Schematic illustrating a two-dissipator gauge symmetry nonreciprocal interaction. Two cavity modes a_1,a_2 are each coupled to waveguides, which are then sent to a beam splitter that depends on the state of a qubit. The dissipation is described by the Liouvillian $\Gamma(\mathcal{D}[\hat{z}_1]+\mathcal{D}[\hat{z}_2])$ [see Eqs. (40) and (42)]. (b) Numerically computed cavity isolation $I^{(A)}_{\{|+_x\rangle,|-_x\rangle\}}(t)$ for dynamics generated by the two-dissipator sum (solid red line) versus a single dissipator $2\Gamma\mathcal{D}[\hat{z}_1]$ (dashed blue curve). The A isolation stays unity in the former case, as expected for any dynamics that is fully nonreciprocal by design [see the discussion on Eq. (40)]. In contrast, individual dissipators from the setup in (a), e.g., $\mathcal{D}[\hat{z}_1]$, are not unidirectional. For illustrative purposes, the numerics plotted here is restricted to the subspace with at most two photons in A, but our result remains valid for a_1, a_2 modes with infinite levels. (c) Entanglement (logarithmic negativity $E_{\mathcal{N}}$) generation by the dissipations, assuming a product initial state $|11\rangle_A \otimes |+_y\rangle_B$. The fully nonreciprocal interactions generated by $\Gamma(\mathcal{D}[\hat{z}_1]+\mathcal{D}[\hat{z}_2])$ creates entanglement (solid red curve), signaling nontrivial influence from the cavity modes to the qubit.

[see Fig. 4(a)]. We take the basic dissipative process to be single-photon loss on each cavity mode, i.e., $\hat{A}_\ell = \hat{a}_\ell$ $(\ell = 1, 2)$. Consider first the setup without the qubit. In this case, the right-hand side of master equation (36) describes a setup where each cavity couples to its own output waveguide, with the two waveguides then being routed through a beam splitter with scattering matrix $\breve{\mathcal{U}}$. The two outputs of the beam splitter are then routed to zero-temperature reservoirs. One clearly sees that the particular form of the beam splitter unitary is immaterial to the two cavity modes: regardless of its form, each cavity mode experiences identical, independent single-photon loss.

We now use our general recipe to make this into a nonreciprocal interaction between the two cavity modes and a qubit, by making the beam splitter a "quantum" beam splitter whose scattering properties depend on qubit operators. Such a qubit-controlled beam splitter can be characterized by a unitary operator $\hat{U}_{\rm qbs}$ on the total cavity-qubit system. We take this operator to depend on both $\hat{\sigma}_z$ and $\hat{\sigma}_x$, with a total system dynamics that is described by

$$\mathcal{L}_2\hat{\rho} = \Gamma\sum_{\ell=1}^{2}\mathcal{D}[\hat{z}_\ell]\hat{\rho}, \qquad \hat{z}_\ell = \hat{U}_{\rm qbs}\hat{a}_\ell(\hat{U}_{\rm qbs})^\dagger, \tag{40}$$

with

$$\hat{U}_{\rm qbs} = e^{i\varphi\hat{\sigma}_z(\hat{a}_1^\dagger\hat{a}_1-\hat{a}_2^\dagger\hat{a}_2)+i\theta\hat{\sigma}_x(\hat{a}_1^\dagger\hat{a}_2+\hat{a}_2^\dagger\hat{a}_1)}. \tag{41}$$

The qubit-controlled beam splitter in Eq. (41) generalizes simpler constructions only involving a single-qubit operator. The latter setups have been discussed theoretically (see, e.g., Refs. [37,64]) and even realized recently in experiment [65]. As we will see, using two noncommuting operators in our example will give us a new dynamical structure that is distinct from the single-dissipator setup in Sec. III A.

The jump operators in Eq. (40) can be explicitly computed to be

$$\hat{z}_1 = e^{-i\varphi\hat{\sigma}_z}(\cos\theta\ \hat{a}_1 - i\sin\theta\ \hat{\sigma}_x\hat{a}_2), \tag{42a}$$

$$\hat{z}_2 = e^{i\varphi\hat{\sigma}_z}(-i\sin\theta\ \hat{\sigma}_x\hat{a}_1 + \cos\theta\ \hat{a}_2). \tag{42b}$$

Each of these dissipators has a nontrivial action on the composite system that in general will correlate the state of the qubit and cavities [see Fig. 4(a)]. For example, the jump operator $\hat{z}_1$ has an amplitude for flipping the state of the qubit (correlated with loss from a_2) and for not flipping the qubit state (correlated with loss from a_1). The situation is reversed for $\hat{z}_2$. In general, neither of these jump operators can be written as a product of a qubit operator times a cavity operator. For simplicity, we assume that $\theta = \varphi = \pi/4$ in what follows, but our results are valid as long as $\sin 2\theta \sin 2\varphi \neq 0$. Note that the latter constraint ensures that Eq. (40) cannot be decomposed into a trivial sum of single nonreciprocal dissipators.

The dissipators in master equation (40) have the same form as those in the general nonreciprocal master equation (38). A direct computation also shows that the unitarity condition of Eq. (39) is satisfied. As a result, Eq. (40) *necessarily* describes a unidirectional interaction from the subsystem-A cavity modes to subsystem B, the qubit. The cavity modes are unaffected by the qubit, and each experience simple loss at rate Γ. In contrast, the qubit remains nontrivially influenced by the cavity modes. It is interesting to ask what this nonreciprocal interaction means physically. To obtain intuition, one can consider the simple case where $\hat{a}_1, \hat{a}_2$ are replaced by c numbers α_1, α_2.

The form of qubit dissipation then crucially depends on the relative phase between the two amplitudes, i.e., the phase of α_1/α_2: if the ratio is real, the two qubit jump operators will be proportional to two unaligned unitaries, and the resulting qubit dynamics simply corresponds to coupling to classically stochastic Hamiltonians [46]. In contrast, if α_1/α_2 is complex, i.e., α_1, α_2 have different phases, then the jump operators will not be proportional to unitaries, generating richer dissipative dynamics for the qubit.

An alternative picture for Eq. (40) is provided by viewing the dynamics as due to an average over stochastic quantum jumps [66]. As we show, this is particularly useful for understanding entanglement generation by the nonreciprocal dynamics. Consider a product initial state, $|\Psi_{AB}\rangle = |11\rangle_A \otimes |\psi_{\rm qb}\rangle_B$ with a generic qubit state $|\psi_{\rm qb}\rangle_B$. With a single quantum jump, the initial state is acted on once by $\hat{z}_1$ or $\hat{z}_2$ [see Eq. (42)]. Because the action on the qubit state by these jump operators is correlated with the action on the two-photon state, the total qubit-cavity state will in general become entangled after the jump, indicating possible entanglement generation in the averaged dynamics. Later in this section, we further provide a concrete example for entanglement generation by Eq. (40).

To help see these properties more explicitly, and to see that the directionality is *not* a property of each dissipator on its own, we can calculate the isolation function of subsystem A [defined via Eqs. (5) and (7)]. We compare the nonreciprocal dynamics as generated by Eq. (40), against the case with just a single dissipator, i.e.,

$$\mathcal{L}_2' \hat{\rho} = 2\Gamma \mathcal{D}[\hat{z}_1]\hat{\rho}, \tag{43}$$

where the jump operator is again given by Eq. (42a). In Fig. 4(b), we plot the conditional subsystem-A isolation $I^{(A)}_{\{|+_x\rangle,|-_x\rangle\}}(t)$, defined as

$$I^{(A)}_{\{|+_x\rangle,|-_x\rangle\}}(t) \equiv 1 - \tfrac{1}{2}\left\| \mathcal{E}^{(A)}_{|+_x\rangle}(t) - \mathcal{E}^{(A)}_{|-_x\rangle}(t) \right\|_\diamond. \tag{44}$$

This quantity measures how sensitive the subsystem-A dynamics is to the initial state of B, when B starts in a σ_x eigenstate $|\pm_x\rangle$. It sets an upper bound on the full A subsystem isolation $I^{(A)}(t)$. As shown in Fig. 4(b), the conditional cavity isolation stays unity at all times for master equation (40). This is as expected, as the generalized unitarity constraint is satisfied. As a result, $I^{(A)}(t)$ should be unity according to Eq. (15). In contrast, the A isolation is significantly smaller than 1 for the case where we only have one of the two required dissipators; see Eq. (43). In this case, A is not isolated from B. This shows concretely that the combined action of both dissipators in Eq. (40) leads to a fully directional dynamics, even though each on its own *does not* mediate a one-way interaction. Note that in this plot we have calculated the isolation functions in a restricted cavity Hilbert subspace with at most two total photons. However, this does not affect the validity of our conclusion, since numerically computed isolation will set an upper bound for the isolation of the bosonic A subsystem with infinite levels. As such, our calculation clearly shows that we cannot get fully nonreciprocal interaction with only one of the dissipators in Eq. (42).

Of course, simply showing that subsystem A is isolated does not indicate a nonreciprocal interaction: we also need to verify that B is influenced by A, and that we have not simply canceled any interaction between the two subsystems. To show that there is indeed a nonreciprocal interaction, in Fig. 4(c) we show that our master equation [Eq. (40)] can generate entanglement between the two subsystems. We show in that figure the time-dependent entanglement, as quantified by the logarithmic negativity, between A and B starting with an initial product state $|11\rangle_A \otimes |+_y\rangle_B$, with $|+_y\rangle_B$ denoting the qubit $\hat{\sigma}_y$ eigenstate. We see that entanglement is generated at intermediate times (red solid curve), even though system A is fully isolated at all times. This indicates that A must be influencing B, and that we have a nontrivial nonreciprocal interaction.

The fact that our dynamics can generate entanglement also leads to other important conclusions. It immediately implies that Eq. (40) cannot be realized via local measurement and feedforward processes, and as such, cannot be rewritten as a sum of single nonreciprocal dissipators, each having form (17). If such a decomposition were possible (see, e.g., Appendix F 2), then our dynamics would be equivalent to a local measurement-plus-feedforward protocol, something that cannot generate entanglement [cf. the discussion below Eq. (25)]. We note that this nonreciprocal entanglement generation is unique to the multidissipator version of our master equation.

Finally, as in the simpler single-dissipator version of our mechanism, deviations from the Markovian limit will also impact the directionality of our interaction; this is discussed in more detail in Appendix C 2.

VI. SUMMARY AND OUTLOOK

In this work, we have introduced and analyzed a new kind of dissipative dynamics that leads to fully nonreciprocal interactions between two quantum systems. The crucial ingredient was a time-local gauge symmetry inherent in any Markovian, Lindblad master equation. Surprisingly, the explicit breaking of time reversal or the use of synthetic gauge fields were not necessary. As such, our new class of directional quantum master equations do not have the form of a standard cascaded quantum master equation.

Nonreciprocal quantum interactions are being actively studied for both their fundamental and practical implications. Our results thus greatly expand the toolbox and class

of interactions available for such studies. In terms of application, we have shown how our interactions can be used for a new kind of dissipative quantum gate; the application to more complex kinds of quantum control (using, e.g., the multidissipator version of our dynamics) could be extremely fruitful. We note that in a very different context, engineered dissipation has been studied theoretically [40,44] and demonstrated experimentally [41–43] as means to realize autonomous quantum error correction (AQEC). Here, dissipative processes are designed to mitigate errors by bringing the system back to a desired code space. While our dissipative nonreciprocity and dissipative gates are very different in nature, it is worth asking whether these ideas could be combined with autonomous error correction for even more robust forms of quantum information processing. In Ref. [32], the general nonreciprocal structure in Eq. (17) is shown to enable a novel AQEC scheme against excitation loss, a dominant error source in bosonic systems. In concatenation with discrete-variable QEC codes, it further offers a promising route towards practical fault-tolerant quantum computation (see Ref. [32] for detail). In another example, cases of nonreciprocal dissipators in Eq. (17) are also considered for achieving passively protected quantum memory [67]. Unlike our work, Lieu *et al.* [67] did not note the general underlying nonreciprocal structure in Eq. (17), nor provide any route to physically realizing the required dissipators. An important open question is under what circumstances such nonreciprocal dissipators would provide a practical advantage in applications; we leave a systematic study to future works.

Our work also has great potential for fundamental studies. For example, it provides a direct way of designing quantum analogues of classical kinetically constrained models that feature directionality (see, e.g., Ref. [68]). Such models could provide a new setting to study glassy dynamics in the quantum regime. It would also be interesting to study our new kind of dissipative interactions in many-body lattice models. Here, our mechanism could be used to construct a class of directional models that are dissipative analogues of closed systems with dynamical gauge fields. The latter is a topic of intense interest in a variety of engineered quantum systems (see, e.g., Refs. [69–72]).

ACKNOWLEDGMENTS

This work is supported by the Army Research Office under Grant No. W911NF-19-1-0380. A.A.C. acknowledges support from the Simons Foundation, through a Simons Investigator award (Grant No. 669487). C.W. is supported by the U.S. Department of Energy, Office of Science, National Quantum Information Science Research Centers, Co-Design Center for Quantum Advantage under Contract No. DE-SC0012704.

APPENDIX A: EXAMPLES OF HAMILTONIAN DYNAMICS THAT IS FULLY RECIPROCAL ACCORDING TO EQ. (10)

In Sec. II in the main text, we introduce a new metric of isolation to quantify the influence of one subsystem on the dynamics of another subsystem, which leads to a general definition of reciprocity and nonreciprocity in the quantum regime. It is interesting to ask what the isolation looks like, and how nonreciprocal it is, if one considers fully coherent (i.e., Hamiltonian) dynamics. In this appendix, we provide two general classes of Hamiltonian dynamics, which can be proven to be reciprocal as per Eq. (10). It is also intriguing to ask if fully Hamiltonian dynamics of generic systems should always be reciprocal by the definition in Eq. (10). While we conjecture that it should be case, in order for the definition of (non)reciprocity in Sec. II to align with the (breaking of) Onsager reciprocity relations, we leave a thorough study to future works.

1. Isolation function of reciprocal dynamics generated by Hamiltonians with local qubit (B) $\mathbb{Z}_2$ symmetry

While our proof can be straightforwardly generalized to Hamiltonian dynamics of a generic bipartite system with an N_A- (N_B-)dimensional subsystem A (B), as long as the total AB Hamiltonian has a local symmetry that is N_B dimensional and nondegenerate within the B subspace, for the sake of clarity, here we focus on bipartite systems AB where B is a single qubit. In this subsection, we explicitly derive the isolation functions for dynamics generated purely by Hamiltonians with a local qubit $\mathbb{Z}_2$ symmetry, and show that any such dynamics must be fully reciprocal by the definition in Eq. (10). We stress that such Hamiltonians, albeit having a constrained form, can be generally nonlinear and interacting.

Consider a generic bipartite system consisting of an N_A-dimensional system as A and a qubit as B. We assume that the system evolves under a Hamiltonian $\hat{H}_{AB}$, where interaction between A and B commutes with the qubit-only Hamiltonian $\hat{H}_{B,0} \equiv \mathrm{Tr}_A \hat{H}_{AB}$. We can thus define, without loss of generality, the eigenstates of $\hat{H}_{B,0}$ as a qubit-$\hat{\sigma}_z$ eigenbasis, and rewrite the system Hamiltonian as

$$\hat{H}_{AB} = \hat{H}_A \otimes \hat{\mathbb{I}}_B + \hat{\xi}_A \otimes \hat{\sigma}_z. \tag{A1}$$

For notational simplicity, it is convenient to rewrite the total Hamiltonian in terms of projectors onto qubit-$\hat{\sigma}_z$ eigenstates as

$$\hat{H}_{AB} = \hat{H}^{(A)}_{\uparrow} \otimes |\uparrow\rangle\langle\uparrow| + \hat{H}^{(A)}_{\downarrow} \otimes |\downarrow\rangle\langle\downarrow|, \tag{A2}$$

where the conditional A Hamiltonians contingent on qubit states $|\sigma\rangle$ ($\sigma = \uparrow, \downarrow$) are given by $\hat{H}^{(A)}_{\uparrow/\downarrow} = \hat{H}_A \pm \hat{\xi}_A$.

Following Eq. (5) in the main text, we can again define the evolution superoperator of A, depending on initial qubit

(B) state $|\phi_i\rangle$, as

$$\mathcal{E}^{(A)}_{|\phi_i\rangle}(t)\hat{\rho}_A \equiv \mathrm{Tr}_B[e^{-i\hat{H}_{AB}t}(\hat{\rho}_A \otimes |\phi_i\rangle_B\langle\phi_i|)e^{i\hat{H}_{AB}t}]. \quad \text{(A3)}$$

Substituting Eq. (A2) into the above equation, the conditional A quantum map can be straightforwardly calculated as

$$\mathcal{E}^{(A)}_{|\phi_i\rangle}(t)\hat{\rho}_A = \sum_{\sigma=\uparrow,\downarrow} |\langle\sigma|\phi_i\rangle|^2 e^{-i\hat{H}^{(A)}_\sigma t}\hat{\rho}_A e^{i\hat{H}^{(A)}_\sigma t}. \quad \text{(A4)}$$

Since the A evolution in this case can be rewritten as a probabilistic mixture of unitary gates, the A isolation can be shown to have a closed analytical form. For convenience, we first introduce the unitary operator $\hat{U}^{(A)}_{\downarrow\uparrow}(t)$ acting on A that describes the "overlap" between the two conditional unitary maps as

$$\hat{U}^{(A)}_{\downarrow\uparrow}(t) = e^{i\hat{H}^{(A)}_\downarrow t}e^{-i\hat{H}^{(A)}_\uparrow t}, \quad \text{(A5)}$$

and define its eigenvalues as $e^{i\phi_\ell}$ ($\ell = 1, 2, \ldots, N_A$). One can thus show that

$$I^{(A)}(t) = 1 - \max_{1\le\ell<m\le N_A}\left|\sin\frac{(\phi_\ell - \phi_m)}{2}\right|. \quad \text{(A6)}$$

Conversely, we can also compute the qubit (B) isolation exactly, since the qubit undergoes a constrained form of dynamics that conserves the $\hat{\sigma}_z$ operator [cf. the system Hamiltonian in Eq. (A2)]. In this case, the qubit experiences a simple phase shift and/or dephasing during the time evolution. More specifically, given initial qubit state $\hat{\rho}_B$ and expanding it using the $\hat{\sigma}_z$ basis, the qubit populations stay constant throughout the evolution. The qubit coherence due to Eq. (A2) can be computed as

$$\begin{aligned}\frac{\langle\uparrow|\mathcal{E}^{(B)}_{|\phi_i\rangle}(t)\hat{\rho}_B|\downarrow\rangle}{\langle\uparrow|\hat{\rho}_B|\downarrow\rangle} &= \mathrm{Tr}(e^{-i\hat{H}^{(A)}_\uparrow t}|\phi_i\rangle_A\langle\phi_i|e^{i\hat{H}^{(A)}_\downarrow t}) \\ &= \mathrm{Tr}(\hat{U}^{(A)}_{\downarrow\uparrow}(t)|\phi_i\rangle_A\langle\phi_i|), \quad \text{(A7)}\end{aligned}$$

where $\hat{U}^{(A)}_{\downarrow\uparrow}$ is again the overlap unitary in Eq. (A5). Because the qubit map now takes a pure-dephasing form, the diamond norm between two such qubit maps can be explicitly derived as

$$\begin{aligned}&||\mathcal{E}^{(B)}_{|\phi_1\rangle}(t) - \mathcal{E}^{(B)}_{|\phi_2\rangle}(t)||_\diamond \\ &\quad = |\mathrm{Tr}[\hat{U}^{(A)}_{\downarrow\uparrow}(t)(|\phi_1\rangle\langle\phi_1| - |\phi_2\rangle\langle\phi_2|)]| \\ &\quad = |\langle\phi_1|\hat{U}^{(A)}_{\downarrow\uparrow}(t)|\phi_1\rangle - \langle\phi_2|\hat{U}^{(A)}_{\downarrow\uparrow}(t)|\phi_2\rangle|. \quad \text{(A8)}\end{aligned}$$

Expanding the states $|\phi_1\rangle, |\phi_2\rangle$ in Eq. (A8) using the eigenbasis of $\hat{U}^{(A)}_{\downarrow\uparrow}$, and optimizing over all A initial states, one can compute the B isolation function $I^{(B)}(t)$ as

$$\begin{aligned}I^{(B)}(t) &\equiv 1 - \frac{1}{2}\max_{|\phi_1\rangle,|\phi_2\rangle\in\mathcal{H}_A}\left\|\mathcal{E}^{(B)}_{|\phi_1\rangle}(t) - \mathcal{E}^{(B)}_{|\phi_2\rangle}(t)\right\|_\diamond \\ &= 1 - \frac{1}{2}\max_{|\phi_1\rangle,|\phi_2\rangle\in\mathcal{H}_A}\left|\langle\phi_1|\hat{U}^{(A)}_{\downarrow\uparrow}(t)|\phi_1\rangle\right. \\ &\quad \left. - \langle\phi_2|\hat{U}^{(A)}_{\downarrow\uparrow}(t)|\phi_2\rangle\right| \\ &= 1 - \max_{1\le\ell<m\le N_A}\left|\sin\frac{(\phi_\ell - \phi_m)}{2}\right|. \quad \text{(A9)}\end{aligned}$$

Comparing the above expression to Eq. (A6), we thus have

$$I^{(A)}(t) = I^{(B)}(t). \quad \text{(A10)}$$

2. Proof of reciprocity for arbitrary Hamiltonian dynamics of two-qubit systems

In this subsection, we restrict to bipartite systems where both A and B are single qubits, and we seek to prove that arbitrary Hamiltonian dynamics of this two-qubit system is reciprocal as per Eq. (10). To start, we note that the isolation function of A (B) is invariant under applications of local unitaries at the input and/or output ports of the quantum channel. More specifically, we consider two generic quantum maps $\mathcal{E}^{(AB)}$ and $\mathcal{F}^{(AB)}$ that are related by the equation

$$\mathcal{E}^{(AB)}\hat{\rho}_{AB} = (\hat{W}_A \otimes \hat{W}_B)(\mathcal{F}^{(AB)}\hat{\rho}'_{AB})(\hat{W}^\dagger_A \otimes \hat{W}^\dagger_B), \quad \text{(A11)}$$

$$\hat{\rho}'_{AB} = \hat{V}_A \otimes \hat{V}_B\hat{\rho}_{AB}\hat{V}^\dagger_A \otimes \hat{V}^\dagger_B. \quad \text{(A12)}$$

Making use of the fact that the diamond norm [cf. Eq. (6)] is invariant under unitary transformations applied before or after the quantum channels [34], one can straightforwardly show that the corresponding isolation functions $I^{(A/B)}(\mathcal{E}^{(AB)})$ and $I^{(A/B)}(\mathcal{F}^{(AB)})$ are also equal for two maps, i.e.,

$$I^{(A/B)}(\mathcal{E}^{(AB)}) = I^{(A/B)}(\mathcal{F}^{(AB)}). \quad \text{(A13)}$$

The isolation functions in the above equation are defined similarly to Eq. (7) for the two channels, and we omit any time variables (if applicable) for notational simplicity.

We next observe that, as pointed out in the main text, dynamics generated by any Hamiltonian symmetric under permutation of A and B is automatically reciprocal by our definition. For two-qubit systems, and assuming no local Hamiltonians, we conclude that dynamics generated by so-called Heisenberg XYZ interactions should be reciprocal. Rewriting the corresponding unitary evolution

superoperator as

$$\mathcal{U}^{(AB)}_{XYZ}\hat{\rho}_{AB} = \hat{U}_{XYZ}\hat{\rho}_{AB}\hat{U}^{\dagger}_{XYZ}, \tag{A14}$$

$$\hat{U}_{XYZ} = \exp\left[-i\left(\sum_{\alpha=x,y,z} J_{\alpha}\hat{\sigma}_{A,\alpha}\hat{\sigma}_{B,\alpha}\right)\right], \tag{A15}$$

we thus have

$$I^{(A)}(\mathcal{U}^{(AB)}_{XYZ}) = I^{(B)}(\mathcal{U}^{(AB)}_{XYZ}). \tag{A16}$$

Finally, we make use of the standard decomposition of any two-qubit rotations into local unitaries and a Heisenberg unitary [73], which can be viewed as a special case of the KAK decomposition of Lie groups. For a generic two-qubit unitary operator $\hat{U}_{AB}$, the decomposition states that there exist two local unitaries $\hat{U}_{A,\ell} \otimes \hat{U}_{B,\ell}$ $(\ell = 1, 2)$ such that the following equality holds:

$$\hat{U}_{AB} = (\hat{U}_{A,1} \otimes \hat{U}_{B,1})\hat{U}_{H}(\hat{U}_{A,2} \otimes \hat{U}_{B,2}) \tag{A17}$$

with

$$\hat{U}_{H} = \exp\left[-i\left(\sum_{\alpha=x,y,z} h_{\alpha}\hat{\sigma}_{A,\alpha}\hat{\sigma}_{B,\alpha}\right)\right]. \tag{A18}$$

Comparing the above equation with Eq. (A11), we see that the unitary map generated by $\hat{U}_{AB}$ has the same isolation functions as that generated by $\hat{U}_{H}$. From Eq. (A16), we can further prove that, for any two-qubit unitary evolution $\mathcal{U}^{(AB)}_{2}\hat{\rho}_{AB} = \hat{U}_{AB}\hat{\rho}_{AB}\hat{U}^{\dagger}_{AB}$, we have

$$I^{(A)}(\mathcal{U}^{(AB)}_{2}) = I^{(B)}(\mathcal{U}^{(AB)}_{2}), \tag{A19}$$

so that any two-qubit Hamiltonian dynamics must be reciprocal.

APPENDIX B: THE ROLE OF MARKOVIANITY IN THE EMERGENCE OF GAUGE SYMMETRY IN A SINGLE-DISSIPATOR LINDBLAD MASTER EQUATION

As discussed in the main text, the unidirectional nature of dynamics generated by the nonlinear dissipator $\mathcal{D}[\hat{A}\hat{U}_{B}]\hat{\rho}$ [see Eq. (17) in the main text] crucially depends on a fundamental gauge symmetry, which is inherent to Lindblad-form QMEs that describe Markovian environments. In this section, we illustrate this connection using a microscopic model for the quantum dissipation.

Let us consider a generic, single Lindblad dissipator of system A, given by

$$\mathcal{L}_{A,1}\hat{\rho} = \Gamma\mathcal{D}[\hat{A}]\hat{\rho}. \tag{B1}$$

Without loss of generality, we could model the dissipation as due to a bosonic microscopic environment consisting of harmonic oscillator modes b_{ℓ}, so that the system-bath Hamiltonian can be written as

$$\hat{H}_{\rm tot} = \hat{H}_{E} + \hat{H}_{\rm SE}, \qquad \hat{H}_{E} = \sum_{\ell}\omega_{\ell}\hat{b}^{\dagger}_{\ell}\hat{b}_{\ell}, \tag{B2}$$

$$\hat{H}_{\rm SE} = \hat{A}\hat{\xi}^{\dagger} + \text{H.c.}, \qquad \hat{\xi} = \sum_{\ell} g^{*}_{\ell}\hat{b}_{\ell}. \tag{B3}$$

The density of states (DOS) function $\mathcal{J}_{0}[\omega]$ of this bosonic environment can be explicitly computed in terms of the interaction picture bath operator $\hat{\xi}(t) = e^{i\hat{H}_{E}t}\hat{\xi}e^{-i\hat{H}_{E}t}$ as

$$\begin{aligned}\mathcal{J}_{0}[\omega] &\equiv \frac{1}{2\pi}\int_{-\infty}^{\infty}\langle[\hat{\xi}(t), \hat{\xi}^{\dagger}(0)]\rangle e^{i\omega t}dt \\ &= \sum_{\ell}|g_{\ell}|^{2}\delta(\omega - \omega_{\ell}).\end{aligned} \tag{B4}$$

In the Markovian limit, the environmental DOS $\mathcal{J}_{0}[\omega]$ reduces to a constant, i.e., we have

$$\mathcal{J}_{0}[\omega] \equiv \Gamma \quad \Longleftrightarrow \quad \langle[\hat{\xi}(t_{1}), \hat{\xi}^{\dagger}(t_{2})]\rangle = \Gamma\delta(t_{1} - t_{2}). \tag{B5}$$

In this limit, we can integrate out the bath dynamics to obtain a standard Lindblad equation, as given by Eq. (B1).

We now perform a standard gauge transformation that shifts the jump operator phase by a time-dependent real value $\theta(t)$, i.e., $\hat{A} \to \hat{A}' = \hat{A}e^{i\theta(t)}$, so that the new interaction picture system-bath interaction becomes

$$\hat{H}'_{\rm SE}(t) = \sum_{\ell}(g_{\ell}\hat{A}\hat{b}^{\dagger}_{\ell}e^{i\omega_{\ell}t + i\theta(t)} + \text{H.c.}). \tag{B6}$$

Mathematically, the gauge symmetry of the Lindblad dissipator can be understood from the fact that the microscopic bath DOS $\mathcal{J}'[\omega]$ stays invariant under the gauge transformation. More specifically, we can rigorously show that

$$\begin{aligned}\mathcal{J}'[\omega] &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{\ell}|g_{\ell}|^{2}e^{i\omega t - i\omega_{\ell}t}e^{-i\theta(t)+i\theta(0)}dt \\ &= \Gamma\int_{-\infty}^{\infty}\delta(t)e^{-i\theta(t)+i\theta(0)}dt \\ &= \Gamma.\end{aligned} \tag{B7}$$

This derivation formally shows that the microscopic origin of the time-local gauge symmetry of the Lindbladian in Eq. (B1) is due to a completely flat bath DOS function, i.e., due to a Markovian environment.

As mentioned, above derivation requires that the bath is perfectly Markovian, so that the bath correlation function $\langle[\hat{\xi}(t), \hat{\xi}^{\dagger}(0)]\rangle$ is proportional to a delta function. We now

discuss an intuitive way to understand the role of Markovianity in the emergence of the time-local gauge symmetry. Noting that the gauge phase $e^{i\theta(t)}$ enters the interaction picture Hamiltonian in exactly the same manner as dynamical phases of the environmental modes $e^{i\omega_\ell t}$, we could formally absorb time dependence of the gauge phase into the definition of the environmental chemical potential, by going to the following rotating frame defined with respect to $\hat{H}_{\rm rot}$:

$$\hat{H}'_E \to \hat{H}''_E = \hat{H}_E - \hat{H}_{\rm rot}, \tag{B8}$$

$$\hat{b}'_\ell \to \hat{b}''_\ell = e^{i\hat{H}_{\rm rot}t}\hat{b}_\ell e^{-i\hat{H}_{\rm rot}t} = \hat{b}_\ell e^{i\theta(t)}. \tag{B9}$$

Here we have

$$\hat{H}_{\rm rot} = -\dot{\theta}(t)\sum_\ell \hat{b}^\dagger_\ell \hat{b}_\ell. \tag{B10}$$

We remark that the system-bath interaction in the rotating frame takes the same form as the original interaction Hamiltonian in Eq. (B3), i.e., we have

$$\hat{H}''_E = \sum_\ell [\omega_\ell + \dot{\theta}(t)]\hat{b}^\dagger_\ell \hat{b}_\ell, \tag{B11}$$

$$\hat{H}''_{\rm SE} = \sum_\ell (g_\ell \hat{A}\hat{b}^\dagger_\ell + {\rm H.c.}). \tag{B12}$$

Noting that the bath Hamiltonian now varies in time, we can formally *define* a time-dependent bath DOS function $\mathcal{J}[\omega;t]$ as

$$\mathcal{J}[\omega;t] \equiv \sum_\ell |g_\ell|^2 \delta(\omega - \omega_\ell - \dot{\theta}(t)), \tag{B13}$$

which can be related to the original bath DOS function $\mathcal{J}_0[\omega]$ in Eq. (B4) as

$$\mathcal{J}[\omega;t] = \mathcal{J}_0[\omega - \dot{\theta}(t)]. \tag{B14}$$

In the Markovian limit we thus have $\mathcal{J}[\omega;t] \equiv \mathcal{J}_0[\omega] = \Gamma$, so that the bath DOS is invariant under generic gauge transformations. For a realistic environment, the bath bandwidth should be finite and hence cannot be perfectly Markovian. In this case, the above analysis would be valid as long as the bath correlation time τ_E is much smaller than the timescale associated with dynamics of the gauge phase, i.e., $\tau_E\dot{\theta}(t) \ll 1$.

APPENDIX C: LEADING-ORDER NON-MARKOVIAN CORRECTIONS TO FULLY NONRECIPROCAL MASTER EQUATIONS DUE TO BROKEN GAUGE SYMMETRY

1. Single-dissipator case

The discussion in the preceding section relates an intrinsic gauge symmetry of any Lindbladian dissipative dynamics to the Markovian nature of the corresponding bath. In this subsection, we focus on Lindbladians with a single dissipator, and examine how the gauge symmetry can be broken if the bath deviates from the Markovian limit. We again start with the interaction picture system-bath Hamiltonian, where the system-bath coupling is modulated in time by a phase factor (which formally acts as a gauge transformation on the jump operator). The Hamiltonian is given by Eq. (B6) as

$$\hat{H}'_{\rm SE}(t) = \hat{A}\hat{\xi}^\dagger e^{i\theta(t)} + {\rm H.c.} = \sum_\ell (g_\ell \hat{A}\hat{b}^\dagger_\ell e^{i\omega_\ell t + i\theta(t)} + {\rm H.c.}). \tag{C1}$$

Assuming that the bosonic bath is in the vacuum state, we can derive a Markovian evolution equation for the system density matrix as

$$\frac{d}{dt}\hat{\rho}(t) = -i[\Sigma(t)\hat{A}^\dagger\hat{A}, \hat{\rho}(t)] + \Gamma_{\rm BR}(t)\mathcal{D}[\hat{A}]\hat{\rho}(t). \tag{C2}$$

Note that the above equation is an example of the so-called Bloch-Redfield equation, which generalizes the standard Lindblad master equation by incorporating effects due to a finite bath correlation time. The Bloch-Redfield equation still assumes that the bath is Gaussian, but allows the bath to be non-Markovian. The first term on the right-hand side of Eq. (C2) is analogous to the Lamb shift, and describes a correction to the coherent system Hamiltonian due to coupling to the environment. The second term takes the same form as the standard Lindblad dissipator, but the dissipator strength can now be negative because we have included effects from a non-Markovian bath.

To compute the coefficients $\Sigma(t)$ and $\Gamma_{\rm BR}(t)$ in Eq. (C2), it is useful to view Eq. (C1) as coupling the system to a new, effective bath operator, whose phase is modified by the gauge phase $e^{i\theta(t)}$. We refer to this new bath as the "gauge transformed bath" in the following discussions. We can thus introduce the two-point correlation function $\mathcal{E}_\theta(t_1,t_2)$ of the new bath operator, which is given by

$$\mathcal{E}_\theta(t_1,t_2) \equiv \langle \hat{\xi}(t_1)\hat{\xi}^\dagger(t_2)\rangle e^{-i\theta(t_1)+i\theta(t_2)} = \sum_\ell |g_\ell|^2 e^{-i\omega_\ell(t_1-t_2)-i\theta(t_1)+i\theta(t_2)}. \tag{C3}$$

It is important to note that, while the original bath is stationary, the new bath set by the gauge phase is generally nonstationary, except for the trivial case where the gauge phase $\theta(t)$ is only a linear function of time. This means that the correlation function $\mathcal{E}_\theta(t_1,t_2)$ in Eq. (C3) would depend on both the time difference $t_1 - t_2$ and the "center of times" $(t_1+t_2)/2$, unless we have $\theta(t) = \theta(0) + t\dot{\theta}(0)$. The Lamb shift coefficient $\Sigma(t)$ can now be related to the

bath correlation function as

$$\Sigma(t) = \mathrm{Im}\int_0^t dt_1 \langle\hat{\xi}(t)\hat{\xi}^\dagger(t_1)\rangle e^{-i\theta(t)+i\theta(t_1)} = \mathrm{Im}\int_0^t dt_1 \mathcal{E}_\theta(t,t_1), \tag{C4}$$

and the dissipator strength $\Gamma_{\mathrm{BR}}(t)$ is in turn given by

$$\Gamma_{\mathrm{BR}}(t) = 2\mathrm{Re}\int_0^t dt_1 \mathcal{E}_\theta(t,t_1). \tag{C5}$$

If the original bath has a finite but short correlation time τ_E (the definition of short will become clear in the discussion that follows), we can compute the leading-order non-Markovian correction in the master equation, which depends on $\dot{\theta}(t)$. We first rewrite the correlation function of the gauge-transformed bath in terms of the autocorrelation function of the old bath $\mathcal{E}(t_1,t_2)$ [setting $\theta(t)\equiv 0$ in Eq. (C3)] as

$$\mathcal{E}_\theta(t_1,t_2) = \mathcal{E}(t_1,t_2)e^{-i\theta(t_1)+i\theta(t_2)}. \tag{C6}$$

Without loss of generality, we assume that the bath has a finite correlation time τ_E, so that the stationary bath autocorrelation function can be written as

$$\mathcal{E}(t_1,t_2) = \mathcal{E}_\theta(t_1,t_2)\big|_{\theta(t)\equiv 0} = g_{\mathrm{eff}}^2 e^{-|t_1-t_2|/\tau_E}, \tag{C7}$$

where $g_{\mathrm{eff}} = \sqrt{\mathcal{E}(0,0)}$ is a real coupling coefficient that characterizes the system-bath coupling strength. If the gauge phase changes much slower than the bath correlation time, i.e., we have

$$\tau_E^2\ddot{\theta}(t) \ll \tau_E\dot{\theta}(t) \ll 1 \quad \text{for all } t, \tag{C8}$$

and if we are interested in system dynamics over timescales that are much longer than the bath correlation time ($t \gg \tau_E$), then we can approximate the integral entering Eqs. (C4) and (C5) as

$$\begin{aligned}\int_0^t dt_1 \mathcal{E}_\theta(t,t_1) &= g_{\mathrm{eff}}^2\int_0^t e^{-(t-t_1)/\tau_E}e^{-i\theta(t)+i\theta(t_1)}dt_1 \\ &\simeq g_{\mathrm{eff}}^2\int_{-\infty}^t e^{-(t-t_1)/\tau_E}e^{-i\dot{\theta}(t)(t-t_1)+i\ddot{\theta}(t)(t-t_1)^2/2}dt_1 \\ &\simeq g_{\mathrm{eff}}^2\tau_E[1 - i\tau_E\dot{\theta}(t) + i\tau_E^2\ddot{\theta}(t) - \tau_E^2\dot{\theta}(t)^2].\end{aligned} \tag{C9}$$

Thus, we obtain first two leading-order contributions to the Lamb shift, and to the time-dependent decay rate, in terms of the small parameter $\tau_E\dot{\theta}(t)$ as

$$\Sigma(t) \simeq -g_{\mathrm{eff}}^2\tau_E^2[\dot{\theta}(t) - \tau_E\ddot{\theta}(t)], \tag{C10}$$

$$\Gamma_{\mathrm{BR}}(t) \simeq 2g_{\mathrm{eff}}^2\tau_E(1 - [\tau_E\dot{\theta}(t)]^2). \tag{C11}$$

We can thus rewrite Eq. (C2) as

$$\frac{d}{dt}\hat{\rho}(t) \simeq ig_{\mathrm{eff}}^2\tau_E^2[\dot{\theta}(t) - \tau_E\ddot{\theta}(t)][\hat{A}^\dagger\hat{A},\hat{\rho}(t)] + 2g_{\mathrm{eff}}^2\tau_E(1 - [\tau_E\dot{\theta}(t)]^2)\mathcal{D}[\hat{A}]\hat{\rho}(t). \tag{C12}$$

It is also convenient to rewrite the above equation in terms of the original bath DOS $\mathcal{J}_0$ given in Eq. (B4). The latter can now be calculated as

$$\begin{aligned}\mathcal{J}_0[\omega] &\equiv \frac{1}{2\pi}\int_{-\infty}^{\infty}\langle[\hat{B}(t),\hat{B}^\dagger(0)]\rangle e^{i\omega t}dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{E}(t,0)e^{i\omega t} \\ &= \frac{1}{\pi}g_{\mathrm{eff}}^2\tau_E,\end{aligned} \tag{C13}$$

so that we have

$$\partial_t\hat{\rho}(t) \simeq i\pi\mathcal{J}_0[0]\tau_E[\dot{\theta}(t) - \tau_E\ddot{\theta}(t)][\hat{A}^\dagger\hat{A},\hat{\rho}(t)] + 2\pi\mathcal{J}_0[0](1 - [\tau_E\dot{\theta}(t)]^2)\mathcal{D}[\hat{A}]\hat{\rho}(t). \tag{C14}$$

2. Generalization to the multidissipator case

The discussion in the previous subsection can be straightforwardly generalized to a Lindbladian with multiple dissipators. In the main text, we stated that the invariance of a generic Lindbladian under time-dependent unitary transformations $\check{\mathcal{U}}(t)$ [60] on the jump operators is closely related to the Markovian nature of the bath. Here, we illustrate this connection by similarly deriving leading-order non-Markovian corrections to general Lindbladians with multiple dissipators. For concreteness, here we consider a general class of microscopic environments that can realize such dissipators in the Markovian limit, but our approach straightforwardly applies to generic environments.

We start by rewriting a general Lindbladian as [see Eq. (36)]

$$\mathcal{L}_A\hat{\rho}_A = \Gamma\sum_{\ell=1}^N\mathcal{D}\left[\sum_{m=1}^N u_{\ell m}(t)\hat{A}_m\right]\hat{\rho}_A, \tag{C15}$$

where the $u_{\ell m}(t)$ are again matrix elements of the N-dimensional complex unitary matrix $\check{\mathcal{U}}(t)$. For the purpose of discussion, we consider a microscopic environment realizing such dissipative dynamics, by coupling the system to

a harmonic oscillator bath with independent bath operators $\hat{\xi}_\ell$. In the interaction picture with respect to the bath-only Hamiltonian $\hat{H}_E$, the system-bath Hamiltonian is given by

$$\hat{H}_{\rm SE}(t) = \sum_{\ell,m=1}^{N} u_{\ell m}(t)\hat{A}_m\hat{\xi}_\ell^\dagger(t) + \text{H.c.}, \tag{C16}$$

where the bath operators satisfy white-noise statistics

$$\langle\hat{\xi}_\ell^{(0)\dagger}(t_1)\hat{\xi}_{\ell'}^{(0)}(t_2)\rangle = 0, \tag{C17}$$

$$\langle\hat{\xi}_\ell^{(0)}(t_1)\hat{\xi}_{\ell'}^{(0)\dagger}(t_2)\rangle = \Gamma\delta_{\ell\ell'}\delta(t_1-t_2). \tag{C18}$$

In the Markovian limit, the resulting system (A modes) dynamics will be invariant under any unitary matrix $\breve{\mathcal{U}}(t)$. To understand the role of Markovianity in the emergence of this time-local generalized gauge symmetry, we again assume that the bath modes have finite but very small correlation times, and we now derive the correction to A system dynamics due to non-Markovian effects. Towards this goal, it is useful to rewrite the system-bath coupling using a new set of bath operators $\hat{C}_\ell$ as

$$\hat{H}_{\rm SE}(t) = \sum_{\ell=1}^{N}\hat{A}_\ell\hat{C}_\ell^\dagger(t) + \text{H.c.}, \quad \hat{C}_\ell(t) = \sum_{m=1}^{N} u_{m\ell}^*(t)\hat{\xi}_m(t). \tag{C19}$$

In contrast to the original stationary bath operators $\hat{\xi}_m(t)$, similar to the single-dissipator case, the new bath operators $\hat{C}_m(t)$ are in general nonstationary due to the presence of the time-dependent coefficients $u_{m\ell}(t)$. As we show, such nonstationarity will in general give rise to nontrivial non-Markovian effects in A dynamics.

Making use of the standard Born-Markov approximation, we can integrate out the bath modes to obtain an effective master equation for system A:

$$\frac{d\hat{\rho}(t)}{dt} = -i\sum_{\ell,m=1}^{N}\Sigma_{\ell m}(t)[\hat{A}_\ell^\dagger\hat{A}_m,\hat{\rho}(t)] + \sum_{\ell,m=1}^{N}\Gamma_{\ell m}(t)\left(\hat{A}_m\hat{\rho}(t)\hat{A}_\ell^\dagger - \frac{1}{2}\{\hat{A}_\ell^\dagger\hat{A}_m,\hat{\rho}(t)\}\right). \tag{C20}$$

The effective Hamiltonian, also known as the Lamb shift term, and the dissipator coefficient matrices are given by

$$\Sigma(t) = -\frac{i}{2}[\mathcal{S}(t) - \mathcal{S}^\dagger(t)], \tag{C21}$$

$$\Gamma(t) = \mathcal{S}(t) + \mathcal{S}^\dagger(t), \tag{C22}$$

$$\mathcal{S}_{\ell m}(t) = \int_0^t dt_1\langle\hat{C}_\ell(t)\hat{C}_m^\dagger(t_1)\rangle. \tag{C23}$$

While our discussion is applicable to generic forms of bath correlators in the small correlation time limit, for concreteness, we assume that they take the diagonal forms

$$\langle\hat{\xi}_\ell^{(0)\dagger}(t_1)\hat{\xi}_{\ell'}^{(0)}(t_2)\rangle = 0, \tag{C24}$$

$$\langle\hat{\xi}_\ell^{(0)}(t_1)\hat{\xi}_{\ell'}^{(0)\dagger}(t_2)\rangle = \Gamma\delta_{\ell\ell'}\frac{1}{2\tau_{E,\ell}}e^{-|t_1-t_2|/\tau_{E,\ell}}. \tag{C25}$$

We can thus compute the master equation coefficients via the bath kernel function $\mathcal{S}_{\ell m}(t)$. For convenience, we introduce the time-dependent generator of $\breve{\mathcal{U}}(t)$ as

$$\breve{\mathcal{G}}(t) = -i\frac{d\breve{\mathcal{U}}(t)}{dt}\breve{\mathcal{U}}^\dagger(t). \tag{C26}$$

Assuming a small bath correlation time $\Gamma\tau_{E,\ell} \ll 1$, we can perturbatively compute the leading-order corrections to $\mathcal{S}_{\ell m}(t)$ as

$$\begin{aligned}\mathcal{S}_{\ell m}(t) &= \sum_{j,j'=1}^{N} u_{j\ell}^*(t)\int_0^t dt_1\langle\hat{\xi}_j(t)\hat{\xi}_{j'}^\dagger(t_1)\rangle u_{j'm}(t_1)\\ &\simeq \frac{\Gamma}{2}\sum_{j,j'=1}^{N} u_{j\ell}^*(t)u_{j'm}(t)[\delta_{jj'} + i\tau_{E,j}g_{jj'}(t)\\ &\quad\times -i\tau_{E,j}^2\dot{g}_{jj'}(t) - \tau_{E,j}^2 g_{ja}(t)g_{aj'}(t)].\end{aligned} \tag{C27}$$

The corresponding master equation (C20) can now be conveniently rewritten in terms of a new jump operator basis,

$$\hat{z}_\ell(t) = \sum_{m=1}^{N} u_{\ell m}(t)\hat{A}_m, \tag{C28}$$

so that we can write the effective master equation as

$$\begin{aligned}\mathcal{L}\hat{\rho}(t) \simeq -i\sum_{\ell,m=1}^{N}\tilde{\Sigma}_{\ell m}(t)[\hat{z}_\ell^\dagger(t)\hat{z}_m(t),\hat{\rho}(t)] + \sum_{\ell,m=1}^{N}\tilde{\Gamma}_{\ell m}(t)\\ \times\left(\hat{z}_m(t)\hat{\rho}(t)\hat{z}_\ell^\dagger(t) - \frac{1}{2}\{\hat{z}_\ell^\dagger(t)\hat{z}_m(t),\hat{\rho}(t)\}\right).\end{aligned} \tag{C29}$$

The Lamb shift term and dissipator coefficients in the new basis can thus be derived as

$$\begin{aligned}\tilde{\Sigma}_{\ell m}(t) &= \tfrac{\Gamma}{4}[(\tau_{E,\ell}+\tau_{E,m})g_{\ell m}(t) - (\tau_{E,\ell}^2+\tau_{E,m}^2)\dot{g}_{\ell m}(t)\\ &\quad + i(\tau_{E,\ell}^2-\tau_{E,m}^2)g_{\ell a}(t)g_{am}(t)],\end{aligned} \tag{C30}$$

$$\begin{aligned}\tilde{\Gamma}_{\ell m}(t) &= \Gamma + \tfrac{\Gamma}{2}[i(\tau_{E,\ell}-\tau_{E,m})g_{\ell m}(t)\\ &\quad - i(\tau_{E,\ell}^2-\tau_{E,m}^2)\dot{g}_{\ell m}(t)\\ &\quad - (\tau_{E,\ell}^2+\tau_{E,m}^2)g_{\ell a}(t)g_{am}(t)].\end{aligned} \tag{C31}$$

Before ending this subsection, we also discuss an intuitive way to understand the local-in-time gauge symmetry $\check{\mathcal{U}}(t)$ in the multidissipator case. Similar to the single-dissipator scenario, the addition of unitary matrix elements $u_{\ell m}(t)$ in the Lindbladian $\mathcal{L}_A\hat{\rho} = \Gamma\sum_{\ell=1}^{N}\mathcal{D}[\sum_{m=1}^{N}u_{\ell m}(t)\hat{A}_m]\hat{\rho}$ [see Eq. (C15)] can be equivalently viewed as the result of shifting the Hamiltonian frequencies of a microscopic harmonic oscillator bath. Let us again consider a microscopic bath realizing the dissipators, with the system-bath Hamiltonian given by

$$\hat{H}_{\rm tot} = \hat{H}_E + \hat{H}_{\rm SE}, \qquad \hat{H}_E = \sum_{\alpha}\omega_\alpha\sum_{\ell=1}^{N}\hat{b}^\dagger_{\ell,\alpha}\hat{b}_{\ell,\alpha}, \tag{C32}$$

$$\hat{H}_{\rm SE}(t) = \sum_{\ell,m=1}^{N}u_{\ell m}(t)\hat{A}_m\hat{\xi}^\dagger_\ell + \text{H.c.}, \qquad \hat{\xi}_\ell = \sum_{\alpha}g^*_\alpha\hat{b}_{\ell,\alpha}. \tag{C33}$$

Note that we choose the bath mode frequencies ω_α and coupling strengths g^*_α to be identical for corresponding modes $\hat{b}_{\ell,\alpha}$ coupled to different system operators $\hat{A}_\ell$. This bath model is not necessarily physically motivated, but it allows a simple interpretation of the time-dependent coupling coefficients $u_{\ell m}(t)$. In fact, we can now transform to a new rotating frame

$$\hat{\rho}_{\rm SE} \to \hat{\rho}'_{\rm SE} = \hat{U}^\dagger(t)\hat{\rho}_{\rm SE}\hat{U}(t), \qquad \hat{U}(t) = \mathcal{T}e^{-i\int_0^t\delta\hat{H}_E(t_1)dt_1}, \tag{C34}$$

so that the time dependence in system-bath couplings is converted into an additional term in the bath-only Hamiltonian as

$$\hat{H}_{\rm SE} \to \hat{H}'_{\rm SE} = \hat{U}^\dagger(t)\hat{H}_{\rm SE}\hat{U}(t) = \sum_{\ell=1}^{N}\hat{A}_\ell\hat{\xi}^\dagger_\ell + \text{H.c.}, \tag{C35}$$

$$\hat{H}_E \to \hat{H}'_E(t) = \hat{H}_E + \delta\hat{H}_E(t). \tag{C36}$$

The correction term $\delta\hat{H}_E(t)$ takes the form of a beam splitter Hamiltonian, and can be written as

$$\delta\hat{H}_E(t) = \sum_{\alpha}\sum_{\ell,m=1}^{N}h_{\ell m}(t)\hat{b}^\dagger_{\ell,\alpha}\hat{b}_{m,\alpha}, \tag{C37}$$

where the beam splitter matrix $h_{\ell m}(t) = [\check{\mathcal{H}}(t)]_{\ell m}$ is related to the unitary matrix $\check{\mathcal{U}}(t)$ via the equation

$$\check{\mathcal{H}}(t) = i\frac{d\check{\mathcal{U}}^\dagger(t)}{dt}\check{\mathcal{U}}(t). \tag{C38}$$

APPENDIX D: RESERVOIR ENGINEERING IMPLEMENTATION OF NONRECIPROCAL INTERACTIONS IN EQ. (17) VIA A UNIDIRECTIONAL WAVEGUIDE

In Sec. IV A in the main text, we state that the nonreciprocal single-dissipator master equation in Eq. (17) can be straightforwardly realized via reservoir engineering, if one also has access to elements explicitly breaking TRS, e.g., an unidirectional waveguide. In this appendix, we provide a detailed discussion about the physical setup in this case and its connection to related experiments [51,52].

Recall the desired dissipator in Eq. (17), as given by $\mathcal{L}_{\rm dir}\hat{\rho} = \Gamma\mathcal{D}[\hat{A}\hat{U}_B]\hat{\rho}$. For concreteness, in this appendix we assume that the A subsystem is a cavity mode with operator $\hat{A} = \hat{a}$ being a bosonic lowering operator, but we note that the scheme discussed below can be generalized to other systems as well. The corresponding Lindbladian is thus

$$\mathcal{L}_{\rm dir}\hat{\rho} = \Gamma\mathcal{D}[\hat{a}\hat{U}_B]\hat{\rho}. \tag{D1}$$

For reasons that will become clear later, it is convenient to rewrite the unitary in terms of a Hermitian generating operator $\hat{E}_B$, i.e.,

$$\hat{U}_B = \exp(-i\hat{E}_B), \tag{D2}$$

where eigenvalues of $\hat{E}_B$ are real and lie in the range $(-\pi,\pi]$. Noting that the definition of unitary $\hat{U}_B$ has a global gauge phase degree of freedom, we can redefine it as $e^{-i\theta_0}\hat{U}_B$ with arbitrary phase θ_0 without affecting system dynamics; we choose this phase such that -1 is not in the spectrum of $\hat{U}_B$, or, equivalently, $\hat{E}_B$ does not contain eigenvalue π.

We now introduce a fully directional coupling (e.g., mediated by a one-way waveguide) from A to a reservoir mode c, as well as a Hamiltonian interaction $\hat{H}_{BC}$ between B and the reservoir that shifts the c frequency. The total system-reservoir (SR) dynamics can thus be described by the master equation

$$\frac{d\hat{\rho}_{\rm SR}}{dt} = -i[\hat{H}_{AC}+\hat{H}_{BC},\hat{\rho}_{\rm SR}] + \mathcal{D}[\sqrt{\Gamma_a}\hat{a} - i\sqrt{\Gamma_c}\hat{c}]\hat{\rho}_{\rm SR}. \tag{D3}$$

Here, Γ_a, Γ_c denote the effective coupling rates between modes a, c and the directional waveguide, respectively. The coupling Hamiltonians $\hat{H}_{AC}$ and $\hat{H}_{BC}$ are now given by

$$\hat{H}_{AC} = \tfrac{1}{2}\sqrt{\Gamma_a\Gamma_c}(\hat{a}^\dagger\hat{c} + \hat{c}^\dagger\hat{a}), \tag{D4}$$

$$\hat{H}_{BC} = \tfrac{1}{2}\lambda\hat{M}_B\hat{c}^\dagger\hat{c}, \tag{D5}$$

where $\hat{M}_B$ is a dimensionless Hermitian operator on B that we determine later. Note that if we ignore B and its coupling to the reservoir mode c, the remaining setup reduces

to the standard cascaded quantum system [see Eq. (1)]. To realize the dissipator in Eq. (D1), we take the limit where c serves as a Markovian reservoir for the AB system. More specifically, this requires $\Gamma_c \gg \Gamma_a$, in which case we can use standard adiabatic elimination techniques [74] to integrate out mode c. We thus obtain an effective master equation of the system AB as

$$\frac{d\hat{\rho}}{dt} = \Gamma_a \mathcal{D}[\hat{a}\hat{U}_{B,\text{eff}}]\hat{\rho}, \tag{D6}$$

$$\hat{U}_{B,\text{eff}} = \frac{\Gamma_c \hat{\mathbb{I}}_B - i\lambda \hat{M}_B}{\Gamma_c \hat{\mathbb{I}}_B + i\lambda \hat{M}_B}. \tag{D7}$$

It is worth stressing that the validity of Eq. (D6) does not depend on having a small coupling between the reservoir and subsystem B, i.e., λ can be comparable or even greater than Γ_c. Hence, we can use this recipe to realize general unitary operators $\hat{U}_B$.

Comparing Eq. (D6) with Eq. (D1), we can choose the a mode coupling rate Γ_a and subsystem-B coupling operator $\hat{M}_B$ in the initial setup, Eq. (D3), such that the effective master equation (D6) realizes the desired dynamics, i.e., we require

$$\Gamma_a = \Gamma, \tag{D8}$$

$$\hat{M}_B = \frac{\Gamma_c}{\lambda} \tan \frac{\hat{E}_B}{2}. \tag{D9}$$

We thus obtain a general recipe, as given by Eq. (D3), that makes use of directional coupling and reservoir engineering techniques to implement a nonreciprocal quantum master equation of the form (D1). For the specific case where B is a single qubit, and assuming that the target unitary is a Z rotation $e^{-i\theta\hat{\sigma}_z/2}$, we further have

$$\hat{M}_B = \frac{\Gamma_c}{\lambda}\left(\tan \frac{\theta}{4}\right)\hat{\sigma}_z. \tag{D10}$$

We note that the physical setup discussed in this appendix is experimentally accessible using, e.g., state-of-the-art superconducting qubit platforms. In fact, in a different context, specific cases of dynamics in Eq. (D3) have been implemented for quantum nondemolition (QND) measurement of itinerant microwave photons [51,52]. In those works, the QND detector structure consists of a cavity dispersively coupled to a qubit. To perform the QND detection, an itinerant microwave field is sent through a circulator and then reflected off the cavity mode. By measuring the qubit phase shift, one can in turn extract the average photon number of the input pulse. Comparing the QND setup to our master equation (D3), the cavity and the qubit would be mapped to the reservoir mode c and subsystem B, respectively. As a result, the external coupling rate of the cavity would correspond to Γ_c, and the cavity-qubit dispersive interaction to Eq. (D5). For the setup discussed in Ref. [52], we can use a qubit in the place of mode a to describe the single-photon source in that work, so that the recipe in Eq. (D3) can be straightforwardly modified to describe the corresponding experimental system. In Ref. [51], the itinerant microwave has a Gaussian pulse shape, which can be formally mimicked via the output of a cavity mode (A) with a time-dependent coupling rate $\Gamma_a(t)$. In both experiments, the unidirectional coupling is achieved by explicit use of a circulator.

Despite aforementioned similarities between the physical setups used in Refs. [51,52] and our model, we note that those previous works did not really utilize the master equation in Eq. (D3) or (D6) to analyze the dynamics of their systems. Our derivation thus reveals a striking new feature of dynamics in those types of setups: in the Markovian limit ($\Gamma_c \gg \Gamma_a$), if one starts from a product state of the A cavity Fock state and a generic qubit (B) state, the qubit steady state in the long-time limit would undergo a *coherent* Z rotation from its initial state, with the phase shift controlled by the photon number of the initial A state. The experimental setup in Ref. [52] can thus be directly used to dissipatively realize steady-state unitary gates (see also Sec. III C).

Finally, we note that the effective master equation (D6) describes an idealized case; in realistic settings, the fidelity of the unitary gate emerging from such a dissipative steady-sate relaxation process will suffer from a range of imperfections, including non-Markovian effects due to a finite reservoir linewidth, nonzero thermal photon population in the reservoir and cavity modes, imperfect preparation of the initial A cavity state, etc. Still, in principle, it is possible to carefully engineer the practical systems to suppress those factors and achieve gate fidelity comparable to typical pulse-based gates. Furthermore, given an experimental system, one can incorporate those imperfections into the theory model in Eq. (D3) and quantitatively examine how they affect the dissipative steady-state gate fidelity (see, e.g., Sec. IV B for a discussion on non-Markovian effects), which can be useful for designing new setups to improve the gate fidelity.

APPENDIX E: THE ROLE OF MARKOVIANITY IN ACHIEVING UNIDIRECTIONAL INTERACTION FOR THE PHYSICAL IMPLEMENTATION IN EQ. (28)

In Sec. IV A in the main text, we discuss physical implementation of the nonreciprocal dissipator $\mathcal{L}_{\text{gate}}\hat{\rho} = \Gamma\mathcal{D}[e^{-i\theta\hat{\sigma}_z/2}\hat{a}]\hat{\rho}$ via an intermediate reservoir mode [see Eq. (28)]. We also state that this setup crucially requires that the reservoir mode is Markovian. In this appendix, we provide a detailed discussion on the role of Markovianity in such physical realizations.

We start with a general setup consisting of a driven-cavity-qubit system coupled to a reservoir mode. In the rotating frame with respect to the drive frequency, the total system dynamics can be described by the master equation $d\hat{\rho}/dt = \mathcal{L}_0\hat{\rho}$, with the Lindbladian given by

$$\mathcal{L}_0\hat{\rho} = -i[\hat{H}_S + \hat{H}_E + \hat{H}_{\rm int}, \hat{\rho}] + \kappa_a\mathcal{D}[\hat{a}]\hat{\rho} + \kappa_c\mathcal{D}[\hat{c}]\hat{\rho}, \tag{E1}$$

$$\hat{H}_S = -\Delta_a\hat{a}^\dagger\hat{a} + \frac{\lambda_a}{2}\hat{\sigma}_z\hat{a}^\dagger\hat{a} + f_{\rm dr}^*(t)\hat{a} + f_{\rm dr}(t)\hat{a}^\dagger, \tag{E2}$$

$$\hat{H}_E = -\Delta_c\hat{c}^\dagger\hat{c}, \qquad \hat{H}_{\rm int} = (J\hat{a}^\dagger\hat{c} + J^*\hat{c}^\dagger\hat{a}) + \frac{\lambda_c}{2}\hat{\sigma}_z\hat{c}^\dagger\hat{c}. \tag{E3}$$

As discussed in Sec. IV A, in the large reservoir linewidth limit, i.e., $\kappa_c \gg |J|$, one can treat the reservoir mode effectively as a Markovian environment for the cavity-qubit system. In this regime, one can follow standard adiabatic elimination procedures to integrate out the reservoir mode and obtain the desired dynamics. To elucidate the role of Markovianity, here we consider instead the regime with small reservoir linewidth $\kappa_c \lesssim |J|$. As we show, the effects due to non-Markovianity now manifest as extra frequency-dependent corrections to the system Langevin equations of motion.

For concreteness, let us first write out the quantum Langevin equations of the system:

$$i\partial_t\hat{a} = \left(-\Delta_a + \frac{\lambda_a}{2}\hat{\sigma}_z - i\frac{\kappa_a}{2}\right)\hat{a} + J\hat{c} - i\sqrt{\kappa_a}\hat{a}_{\rm in} + f_{\rm dr}(t), \tag{E4}$$

$$i\partial_t\hat{c} = \left(-\Delta_c + \frac{\lambda_c}{2}\hat{\sigma}_z - i\frac{\kappa_2}{2}\right)\hat{c} + J^*\hat{a} - i\sqrt{\kappa_c}\hat{c}_{\rm in}, \tag{E5}$$

$$i\partial_t\hat{\sigma}_- = (\lambda_a\hat{a}^\dagger\hat{a} + \lambda_c\hat{c}^\dagger\hat{c})\hat{\sigma}_-. \tag{E6}$$

However, we note that similar analysis can be carried out at the level of, e.g., the system master equation as well.

While our discussion can be straightforwardly generalized to a generic initial state, for convenience, we assume that the cavity mode starts in a coherent state $|\alpha_0\rangle$, i.e., with amplitude α_0, and the reservoir is in the vacuum state at initial time $t = t_0$. Thus, the system dynamics can be fully determined by solving the following set of linear equations for the cavity and reservoir mode amplitudes $\bar{a}_\sigma$ and $\bar{c}_\sigma$ ($\sigma = \uparrow, \downarrow$):

$$i\partial_t\bar{a}_\sigma = \left(-\Delta_a + \frac{\lambda_a}{2}\sigma_z - i\frac{\kappa_a}{2}\right)\bar{a}_\sigma + J\bar{c}_\sigma + f_{\rm dr}(t), \tag{E7a}$$

$$i\partial_t\bar{c}_\sigma = \left(-\Delta_c + \frac{\lambda_c}{2}\sigma_z - i\frac{\kappa_c}{2}\right)\bar{c}_\sigma + J^*\bar{a}_\sigma, \tag{E7b}$$

with $\sigma_z = \pm 1$ corresponding to $\sigma = \uparrow, \downarrow$. The qubit coherence function can thus be computed by solving the differential equation

$$\frac{1}{\langle\hat{\sigma}_-(t)\rangle}\frac{d\langle\sigma_-(t)\rangle}{dt} = -i\lambda_a\bar{a}_\uparrow\bar{a}_\downarrow^* - i\lambda_c\bar{c}_\uparrow\bar{c}_\downarrow^*. \tag{E8}$$

We can formally integrate out the reservoir mode by transforming to Fourier space and eliminating $\bar{c}_\sigma[\omega]$ from the equations, using the relation

$$\bar{c}_\sigma[\omega] = \frac{J^*}{\omega + \Delta_c - \lambda_c\sigma_z/2 + i\kappa_c/2}\bar{a}_\sigma[\omega]. \tag{E9}$$

For notational convenience, we can rewrite the reservoir amplitude in terms of the reservoir mode susceptibility function $\chi_{c,\sigma}^{(0)}[\omega]$ as

$$\bar{c}_\sigma[\omega] = J^*\chi_{c,\sigma}^{(0)}[\omega]\bar{a}_\sigma[\omega], \tag{E10}$$

$$\chi_{c,\sigma}^{(0)}[\omega] \equiv \frac{1}{\omega + \Delta_c - \lambda_c\sigma_z/2 + i\kappa_c/2}. \tag{E11}$$

As a result, we obtain

$$\begin{aligned}\bar{a}_\sigma[\omega] &= \frac{1}{\omega + \Delta_a - \lambda_a\sigma_z/2 + i\kappa_a/2 - |J|^2/(\omega + \Delta_c - \lambda_c\sigma_z/2 + i\kappa_c/2)}f_{\rm dr}[\omega] \\ &= \frac{1}{\omega + \Delta_a - \lambda_a\sigma_z/2 + i\kappa_a/2 - |J|^2\chi_{c,\sigma}^{(0)}[\omega]}f_{\rm dr}[\omega].\end{aligned} \tag{E12}$$

Without loss of generality, and for illustrative purposes, we assume that the drive is resonant with cavity mode a, and that the local cavity loss is negligible, so that we have $\Delta_a = \kappa_a = 0$. In this case, we can further simplify the Fourier-space cavity amplitudes as

$$\bar{a}_\sigma[\omega] = \frac{1}{\omega - \lambda_a\sigma_z/2 - |J|^2\chi_{c,\sigma}^{(0)}[\omega]}f_{\rm dr}[\omega]. \tag{E13}$$

The above equation allows one to express the cavity mode linear response susceptibilities $\chi_{a,\sigma}[\omega]$ as dressed by the coupling to the reservoir mode:

$$\bar{a}_\sigma[\omega] = \chi_{a,\sigma}[\omega] f_{\rm dr}[\omega], \tag{E14}$$

$$\chi_{a,\sigma}[\omega] = \left(\omega - \frac{\lambda_a}{2}\sigma_z - |J|^2 \chi^{(0)}_{c,\sigma}[\omega]\right)^{-1}. \tag{E15}$$

Noting that the phase of tunnel coupling rate J does not affect linear response susceptibilities of the cavity, we can use a local gauge transformation on the cavity mode to make J completely real. Therefore, without loss of generality, we assume a real coupling constant J between the reservoir and cavity modes hereafter.

Making use of the cavity susceptibilities in Eq. (E15), we can define the self-energy of the cavity mode as $\mathcal{E}_{a,\sigma_z}[\omega] \equiv \omega - (\chi_{a,\sigma}[\omega])^{-1}$, so that we have

$$\bar{a}_\sigma[\omega] = \frac{1}{\omega - \mathcal{E}_{a,\sigma_z}[\omega]} f_{\rm dr}[\omega]. \tag{E16}$$

The cavity mode self-energy can now be conveniently written in terms of qubit-independent and qubit-dependent contributions, i.e.,

$$\mathcal{E}_{a,\sigma_z}[\omega] = \frac{\omega + \Delta_c + i\kappa_c/2}{(\omega + \Delta_c + i\kappa_c/2)^2 - (\lambda_c/2)^2}|J|^2 + \left(\frac{\lambda_a}{2} + \frac{|J|^2}{(\omega + \Delta_c + i\kappa_c/2)^2 - (\lambda_c/2)^2}\frac{\lambda_c}{2}\right)\sigma_z. \tag{E17}$$

Since we are interested in dynamics at timescales much slower than reservoir correlation time $\tau_c \sim \kappa_c^{-1}$, we can choose a specific λ_a to minimize the qubit-state-dependent term in Eq. (E17) at low frequencies as

$$\lambda_a = -{\rm Re}\left[\frac{|J|^2}{(\Delta_c + i\kappa_c/2)^2 - (\lambda_c/2)^2}\right]\lambda_c. \tag{E18}$$

If the cavity mode is resonant with the reservoir mode, i.e., if $\Delta_c = \Delta_a = 0$, the qubit-dependent term in the self-energy [Eq. (E17)] allows perfect cancelation in the stationary limit ($\omega = 0$). In contrast, for transient dynamics at fast frequencies ω, the cavity self-energy can have nontrivial dependence on qubit states, even in the resonant limit.

Let us examine such frequency-dependent corrections in more detail. For convenience, we can rewrite the self-energy in terms of reservoir linewidth κ_c and effective parameters $\theta_{\rm eff}$ and $\Gamma_{\rm eff}$ that describe the stationary limit dissipation, i.e., letting

$$\lambda_c = \kappa_c \tan\frac{\theta_{\rm eff}}{2}, \qquad \Delta_c = 0, \tag{E19}$$

$$\Gamma_{\rm eff} = J^2\frac{4\kappa_c}{\kappa_c^2 + \lambda_c^2} \quad \Longrightarrow \quad J^2 = \frac{1}{4}\Gamma_{\rm eff}\kappa_c \sec^2\frac{\theta_{\rm eff}}{2}, \tag{E20}$$

$$\lambda_a = J^2\frac{4\lambda_c}{\kappa_c^2 + \lambda_c^2} = \Gamma_{\rm eff}\tan\frac{\theta_{\rm eff}}{2}, \tag{E21}$$

so that the self-energy function can be reformulated as

$$\mathcal{E}_{a,\sigma_z}[\omega] = \frac{\Gamma_{\rm eff}}{2}\frac{(\omega + i\kappa_c/2)(\kappa_c/2)\sec^2(\theta_{\rm eff}/2) + \omega(\omega + i\kappa_c)\tan(\theta_{\rm eff}/2)\sigma_z}{(\omega + i\kappa_c/2)^2 - ([\kappa_c/2]\tan[\theta_{\rm eff}/2])^2}. \tag{E22}$$

For the sake of discussion, we can use a representative value for the effective phase shift $\theta_{\rm eff} = \pi/2$, so that above expression further simplifies to

$$\mathcal{E}_{a,\sigma_z}[\omega] = \frac{\Gamma_{\rm eff}}{2}\frac{(\omega + i\kappa_c/2)\kappa_c + \omega(\omega + i\kappa_c)\sigma_z}{\omega(\omega + i\kappa_c) - \kappa_c^2/2}. \tag{E23}$$

The Markovian limit amounts to requiring that the self-energy scale near resonance, i.e., $\Gamma_{\rm eff}$, is much smaller than the frequency range over which $\mathcal{E}_{a,\sigma_z}[\omega]$ significantly changes. The latter frequency scale is in turn set by reservoir mode linewidth κ_c. In the limit where $\Gamma_{\rm eff} \ll \kappa_c$, we can expand the self-energy function in the vicinity of $\omega = 0$, so that we obtain

$$\mathcal{E}_{a,\sigma_z}[\omega] \simeq -i\frac{\Gamma_{\rm eff}}{2}\left(1 + 2\frac{\omega}{\kappa_c}\sigma_z\right). \tag{E24}$$

The second term in the parentheses in Eq. (E24) represents leading-order non-Markovian corrections, which describe qubit backaction to the cavity and cause deviation from the full-nonreciprocity limit.

It is also interesting to consider qubit dynamics in this setup. Because of the nonlinear dispersive coupling, the qubit dynamics can be more complicated in frequency

space: taking the Fourier transform of Eq. (E8), we have

$$\omega \ln\langle\sigma_-[\omega]\rangle = \int_{-\infty}^{+\infty} d\omega_1 (\lambda_a \bar{a}_\uparrow[\omega_1]\bar{a}^*_\downarrow[\omega-\omega_1] + \lambda_c \bar{c}_\uparrow[\omega_1]\bar{c}^*_\downarrow[\omega-\omega_1]). \tag{E25}$$

We can again substitute Eq. (E9) into this equation to obtain

$$\omega \ln\langle\sigma_-[\omega]\rangle = \int_{-\infty}^{+\infty} d\omega_1 \Lambda_a[\omega_1;\omega]\bar{a}_\uparrow[\omega_1]\bar{a}^*_\downarrow[\omega-\omega_1], \tag{E26}$$

$$\Lambda_a[\omega_1;\omega] = \lambda_a + \lambda_c |J|^2 \chi^{(0)}_{c,\uparrow}[\omega]\chi^{(0)}_{c,\downarrow}[\omega-\omega_1]. \tag{E27}$$

In the Markovian limit, the cavity amplitude is approximately independent of the qubit state, so that we have $\bar{a}_\uparrow(t) \simeq \bar{a}_\downarrow(t)$. In this limit, the first term on the right-hand side of Eq. (E27), which is given by λ_a, leads to a pure phase shift on the qubit, while the second term can induce both a phase shift and dephasing in qubit dynamics.

APPENDIX F: PROPERTIES OF MULTIPLE-DISSIPATOR GENERALIZATION OF GAUGE-SYMMETRY NONRECIPROCAL LINDBLADIANS IN EQ. (37)

In Sec. V in the main text, we introduce the multidissipator generalization of gauge-symmetry nonreciprocal Lindbladians, which take the form [cf. Eq. (37)]

$$\mathcal{L}_{\text{multi}}\hat{\rho} = \Gamma \sum_{\ell=1}^{N} \mathcal{D}\Big[\sum_{m=1}^{N} \hat{u}_{\ell m}\hat{A}_m\Big]\hat{\rho}. \tag{F1}$$

In analogy to the classical gauge symmetry (36), the B operators $\hat{u}_{\ell m}$ can be thought of as operator-valued matrix elements of a generalized unitary, which acts on the composite linear space $\mathbb{C}^N \otimes \mathcal{H}_B$ between the *dissipator* space $\mathbb{C}^N$ of the $\hat{A}_\ell$ operators and the B system Hilbert space. In this appendix, we provide more details about the general structure of this generalized unitary. Based on this construction, we also discuss a couple of typical examples illustrating the connections and differences between the multi- and single-dissipator nonreciprocal Lindbladians.

1. Connection to a generalized unitary operator

To motivate the generalized unitary, we again start with a multidissipator Lindbladian acting only on system A, as given by $\mathcal{L}_A\hat{\rho}_A = \Gamma \sum_{\ell=1}^{N} \mathcal{D}[\hat{A}_\ell]\hat{\rho}_A$. As noted in the main text, this Lindbladian is invariant under a generic unitary transformation on the jump operators $\hat{A}_\ell$,

$$\mathcal{L}_A\hat{\rho}_A = \Gamma \sum_{\ell=1}^{N} \mathcal{D}\Big[\sum_{m=1}^{N} u_{\ell m}\hat{A}_m\Big]\hat{\rho}_A, \tag{F2}$$

where the $u_{\ell m}$ are matrix elements of an N-dimensional complex unitary matrix $u_{\ell m} = (\check{\mathcal{U}})_{\ell m}$. We can also write the unitary matrix explicitly in terms of its Hermitian generator $\check{\mathcal{H}}$, i.e., $\check{\mathcal{U}} = \exp(-i\check{\mathcal{H}})$. For the discussion that follows, it is convenient to rewrite the Hermitian matrix $\check{\mathcal{H}}$ as a sum of N^2 basis matrices $\check{\mathcal{E}}_{\ell m}$ as

$$\check{\mathcal{H}} = \sum_{\ell,m=1}^{N} h_{\ell m}\check{\mathcal{E}}_{\ell m}. \tag{F3}$$

Here, $\check{\mathcal{E}}_{\ell m} \equiv \mathbf{e}_\ell\,(\mathbf{e}_m)^\dagger$ are formed by outer products between N basis vectors $\mathbf{e}_\ell$ of the linear space $\mathbb{C}^N$, so that we have $\check{\mathcal{E}}^\dagger_{\ell'\ell}\check{\mathcal{E}}_{m'm} = \delta_{\ell' m'}\check{\mathcal{E}}_{\ell m}$.

We now turn to the nonreciprocal Lindbladian given by Eq. (F1). In analogy to the gauge symmetry described in Eq. (F2), we can generate B operators $\hat{u}_{\ell m}$ via a generalized unitary operator acting on the composite linear space $\mathbb{C}^N \otimes \mathcal{H}_B$ as

$$\hat{u}_{\ell m} = (e^{-i\hat{H}_{\text{gen}}})_{\ell m}, \tag{F4}$$

$$\hat{H}_{\text{gen}} = \sum_{j,j'=1}^{N} \check{\mathcal{E}}_{jj'}\hat{h}_{jj'}, \tag{F5}$$

where the $\hat{h}_{jj'}$ are operators acting on B satisfying $\hat{h}_{jj'} = \hat{h}^\dagger_{j'j}$. One can further show that the unitarity conditions $\sum_\ell \hat{u}^\dagger_{\ell m}\hat{u}_{\ell m'} = \delta_{mm'}\hat{\mathbb{I}}_B$ [see Eq. (39)] hold if and only if the $\hat{u}_{\ell m}$ operators can be rewritten as Eq. (F4). If Eq. (F4) holds, as mentioned in the main text, one can exactly trace out system B to obtain a closed master equation acting on system A as $d\hat{\rho}_A/dt = \Gamma \sum_\ell \mathcal{D}[\hat{A}_\ell]\hat{\rho}_A$, so that the multimode master equation (F1) is again unidirectional.

2. Example cases of multidissipator nonreciprocity equivalent or inequivalent to an incoherent sum of single fully directional dissipators

As discussed in the main text, Eq. (F1) describes a much more general class of fully nonreciprocal dynamics as compared to the single-dissipator case [see Eq. (17)]. Here we discuss in detail conditions for when the former multidissipator dynamics can or cannot be rewritten as an incoherent sum of unidirectional dissipators, i.e., if it is

possible to write an equality relation of the form

$$\mathcal{L}_{\text{multi}}\hat{\rho} \stackrel{?}{=} \Gamma \sum_{\ell} \mathcal{D}[\hat{U}_{B,\ell}\hat{A}'_{\ell}]\hat{\rho}. \tag{F6}$$

While it is difficult to comprehensively characterize all the possible scenarios where such an equivalence relation exists, here we discuss two independently sufficient conditions for Eq. (F6) to hold. The first case can be easily understood in terms of the generalized Hermitian generator in Eq. (F5). More specifically, if we can diagonalize it using a new local A basis as defined by $\check{\mathcal{F}}_{\ell\ell'} = \sum_{j,j'=1}^{N} v_{\ell j'} \check{\mathcal{E}}_{j'j} v^{*}_{\ell' j}$, so that we have

$$\hat{H}_{\text{gen}} = \sum_{j,j'=1}^{N} \check{\mathcal{E}}_{jj'}\hat{h}_{jj'} = \sum_{\ell=1}^{N} \check{\mathcal{F}}_{\ell\ell}\hat{\Phi}_{B,\ell}, \tag{F7}$$

where $v_{\ell m} = (\check{\mathcal{V}})_{\ell m}$ are elements of a unitary matrix, then one can use a few lines of algebra to show that Eq. (F1) in this case is equivalent to a sum of single fully nonreciprocal dissipators. More specifically, we obtain

$$\mathcal{L}_{\text{multi}}\hat{\rho} = \Gamma \sum_{\ell=1}^{N} \mathcal{D}\left[e^{-i\hat{\Phi}_{B,\ell}} \sum_{m=1}^{N} v^{*}_{\ell m}\hat{A}_m\right]\hat{\rho}. \tag{F8}$$

Note that the right-hand side of Eq. (F8) is fully equivalent to the starting Lindbladian, but now each dissipator by itself implements a unidirectional interaction from A to B.

The second case is if we cannot diagonalize the generator in Eq. (F5) [and hence the generalized unitary; see Eq. (F4)] via a local basis change on A operators, but the B system only has a single qubit, and all qubit operators $\hat{u}_{\ell m}$ commute with each other. For such Lindbladians, we can again reformulate it as a sum of single nonreciprocal dissipators via a local linear transformation on $\hat{A}_{\ell}$ jump operators, so that Eq. (F6) holds. To see this, we first note that the total system dynamics is easily solvable by jointly diagonalizing $\hat{u}_{\ell m}$ via a local B basis, which we assume to be the $\hat{\sigma}_z$ basis for simplicity, so that time evolution under $\mathcal{L}_{\text{multi}}$ will conserve excitations in that basis. More concretely, the generalized unitary can now be decomposed using projectors $\hat{P}_{\uparrow(\downarrow)}$ onto $\hat{\sigma}_z$ eigenstates of subsystem B as

$$e^{-i\sum_{j,j'=1}^{N} \check{\mathcal{E}}_{jj'}\hat{h}_{jj'}} = \check{\mathcal{U}}_{\uparrow}\hat{P}_{\uparrow} + \check{\mathcal{U}}_{\uparrow}\hat{P}_{\downarrow}. \tag{F9}$$

Again, we can transform the local A basis to $\check{\mathcal{U}}'_{\uparrow(\downarrow)} = \check{\mathcal{W}}\check{\mathcal{U}}_{\uparrow(\downarrow)}\check{\mathcal{W}}^{\dagger}$ via a unitary matrix $\check{\mathcal{W}}$; for the jump operators in this new dissipator frame to be unidirectional, we require that $\check{\mathcal{U}}'_{\uparrow} = \exp(-i\check{\mathcal{D}})\check{\mathcal{U}}'_{\downarrow}$, where $(\check{\mathcal{D}})_{\ell m} = \delta_{\ell m} d_{\ell}$ denotes a real diagonal matrix. This relation can be realized by transforming to the eigenbasis of $\check{\mathcal{U}}_{\uparrow}\check{\mathcal{U}}^{\dagger}_{\downarrow}$, so that we have

$$\mathcal{L}_{\text{multi}}\hat{\rho} = \Gamma \sum_{\ell=1}^{N} \mathcal{D}[e^{-id_{\ell}\hat{\sigma}_z/2}\hat{A}'_{\ell}]\hat{\rho}. \tag{F10}$$

As a concrete example, we consider A consisting of two bosonic modes. Using $\check{\mathcal{Z}}$ and $\check{\mathcal{X}}$ to denote Pauli matrices acting on $\mathbb{C}^2$, we focus on the dissipator generated by the generalized unitary operator

$$\hat{u}_{\ell m} = [e^{-i\theta(\check{\mathcal{Z}}\hat{\sigma}_z\cos\varphi + \check{\mathcal{X}}\hat{\mathbb{I}}_B\sin\varphi)}]_{\ell m}. \tag{F11}$$

The corresponding system Lindbladian is given by

$$\begin{aligned}\mathcal{L}\hat{\rho} = {} & \Gamma\mathcal{D}[\hat{a}_1(\cos\theta - i\hat{\sigma}_z\cos\varphi\sin\theta) - i\hat{a}_2\sin\theta\sin\varphi]\hat{\rho} \\ & + \Gamma\mathcal{D}[\hat{a}_2(\cos\theta + i\hat{\sigma}_z\cos\varphi\sin\theta) \\ & - i\hat{a}_1\sin\theta\sin\varphi]\hat{\rho}.\end{aligned} \tag{F12}$$

In this case $\check{\mathcal{U}}_{\uparrow(\downarrow)} = e^{-i\theta(\pm\check{\mathcal{Z}}\cos\varphi + \check{\mathcal{X}}\sin\varphi)}$, and we can generally rewrite the Lindbladian into the form (F10). For the specific case with $\theta = \pi/2$, the relevant nonlocal basis simplifies to $\hat{a}_{y,\pm} = (\hat{a}_1 \pm i\hat{a}_2)/\sqrt{2}$, so that we have

$$\mathcal{L}\hat{\rho} = \frac{\Gamma}{2}\mathcal{D}[e^{i(\pi/2-\varphi)\hat{\sigma}_z}\hat{a}_{y,+}]\hat{\rho} + \frac{\Gamma}{2}\mathcal{D}[e^{i(-\pi/2+\varphi)\hat{\sigma}_z}\hat{a}_{y,-}]\hat{\rho}. \tag{F13}$$

More generally, if we cannot diagonalize the generator in Eq. (F5) (and hence the generalized unitary) via a local basis change on A operators, and if $\hat{u}_{\ell m}$ operators do not commute or B has a more complicated level structure than a qubit, then there are cases where it is impossible to write the Lindbladian $\mathcal{L}_{\text{multi}}$ as Eq. (F6), i.e., an incoherent sum of fully nonreciprocal dissipators. For example, we can again consider two bosonic modes as subsystem A, where the generalized unitary is given by

$$\hat{u}_{\ell m} = [e^{-i(\varphi\check{\mathcal{Z}}\hat{\sigma}_z + \theta\check{\mathcal{X}}\hat{\sigma}_x)}]_{\ell m}. \tag{F14}$$

The corresponding system Lindbladian can be written as

$$\begin{aligned}\mathcal{L}\hat{\rho} = {} & \Gamma\mathcal{D}[e^{-i\varphi\hat{\sigma}_z}\hat{a}_1\cos\theta - i\hat{\sigma}_x e^{i\varphi\hat{\sigma}_z}\hat{a}_2\sin\theta]\hat{\rho} \\ & + \Gamma\mathcal{D}[e^{i\varphi\hat{\sigma}_z}\hat{a}_2\cos\theta - i\hat{\sigma}_x e^{-i\varphi\hat{\sigma}_z}\hat{a}_1\sin\theta]\hat{\rho},\end{aligned} \tag{F15}$$

which reproduces Eqs. (40) and (42) in the main text. It is interesting to note that if we were to ignore all $\hat{\sigma}_x$ operators in Eq. (F15), the resulting Lindbladian would still be fully nonreciprocal, but could now be rewritten as $\Gamma(\mathcal{D}[e^{-i\varphi\hat{\sigma}_z}\hat{a}_1] + \mathcal{D}[e^{i\varphi\hat{\sigma}_z}\hat{a}_2])\hat{\rho}$, which would be equivalent to a sum of single unidirectional dissipators. The inclusion of $\hat{\sigma}_x$ in the jump operators would not affect local

dynamics of system A; however, those $\hat{\sigma}_x$ operators matter for evolution of correlations between A and B subsystems. As discussed in Sec. V, those correlations can be highly nonclassical and even result in entanglement generation between A and B.

3. Steady state of multidissipator unidirectional dynamics with bosonic lowering operators as $\hat{A}_\ell$

It is interesting to note that if subsystem-A operators $\hat{A}_\ell$ are bosonic lowering operators of cavity modes a_ℓ, i.e., $\hat{A}_\ell = \hat{a}_\ell$, and if A is initialized in a Fock state where at most one cavity mode has nonzero photon(s), then we can further analytically derive the long-time limit of the quantum map generated by $\mathcal{L}_{\text{multi}}$. To see this, we first expand master equation (37) [i.e., Eq. (F1)] and rewrite it as sum of quantum jump and no-jump contributions. Making use of the unitarity conditions in Eq. (39), the equation can be simplified as

$$\mathcal{L}_{\text{multi}}\hat{\rho} = \Gamma \sum_{m,m'=1}^{N} \hat{A}_m \mathcal{E}_{mm'}(\hat{\rho})\hat{A}_{m'}^\dagger - \frac{\Gamma}{2}\left\{\sum_{n=1}^{N}\hat{A}_n^\dagger\hat{A}_n, \hat{\rho}\right\}, \tag{F16}$$

$$\mathcal{E}_{mm'}(\hat{\rho}) = \sum_{j=1}^{N} \hat{u}_{jm}\hat{\rho}\hat{u}_{jm'}^\dagger. \tag{F17}$$

If $\hat{A}_m = \hat{a}_m$, it is straightforward to show that

$$\lim_{t\to\infty} e^{\mathcal{L}_{\text{multi}}t}[(\hat{a}_m^\dagger)^\ell|0\rangle\langle 0|\hat{a}_m^\ell \otimes \hat{\rho}_{B,i}] = (\mathcal{E}_{mm})^\ell(\hat{\rho}_{B,i}). \tag{F18}$$

Thus, for general $\mathcal{L}_{\text{multi}}$ that cannot be rewritten as a sum of single nonreciprocal dissipators, the long-time limit of dynamics within subsystem B generated by the unidirectional interaction will in general be dissipative, in contrast to the single-dissipator case in Eq. (17) (cf. Sec. III C).

APPENDIX G: GAUGE-SYMMETRY NONRECIPROCITY BETWEEN TWO BOSONIC MODES

As discussed in the main text, our recipe for building nonreciprocal QMEs is generally applicable to a broad range of systems. Here we consider an alternative type of system that hosts this physics; more specifically, we look at a nonreciprocal dissipator between two cavity modes a and b, as described by the Lindblad master equation

$$\frac{d\hat{\rho}}{dt} = \mathcal{L}_{\text{bos}}\hat{\rho} = \Gamma\mathcal{D}[e^{-i\theta\hat{x}_b/2}\hat{a}]\hat{\rho}, \tag{G1}$$

where $\hat{x}_b$ denotes the standard quadrature operator of the b mode,

$$\hat{x}_b = \frac{1}{\sqrt{2}}(\hat{b} + \hat{b}^\dagger). \tag{G2}$$

We can write out the corresponding Ito quantum stochastic differential equations [74] for a generic system operator $\hat{M}$ as

$$\frac{d\hat{M}}{dt} = -[\hat{M}, e^{i\theta\hat{x}_b/2}\hat{a}^\dagger]\left[\frac{\Gamma}{2}e^{-i\theta\hat{x}_b/2}\hat{a} + \sqrt{\Gamma}\hat{a}_{\text{in}}\right] + \left[\frac{\Gamma}{2}e^{i\theta\hat{x}_b/2}\hat{a}^\dagger + \sqrt{\Gamma}\hat{a}_{\text{in}}^\dagger\right][\hat{M}, e^{-i\theta\hat{x}_b/2}\hat{a}]. \tag{G3}$$

Equation (G1) takes the general form of nonreciprocal master equations via gauge symmetry discussed in the main text [see Eq. (17)]. It is thus straightforward to show that time evolution of the expectation value of any local operator acting on the a cavity is closed within the a mode, i.e., Eq. (G1) describes a unidirectional coupling from mode a to b. The reverse is, as expected, not true: dynamics of the second cavity mode b in general depends on cavity mode a. For example, we can derive the equation of motion for the b mode momentum average $\langle\hat{p}_b\rangle$ as

$$\frac{d\langle\hat{p}_b\rangle}{dt} = -\gamma\frac{\theta}{2}\langle\hat{a}^\dagger\hat{a}\rangle. \tag{G4}$$

This equation can be intuitively understood in terms of the measurement-and-feedforward picture (cf. Sec. III E): the dissipator $\mathcal{D}[e^{-i\theta\hat{x}_b/2}\hat{a}]\hat{\rho}$ describes a process where cavity mode a is weakly coupled to a photodetector, and every time a photon is detected, we apply a unitary transformation $e^{-i\theta\hat{x}_b/2}$ to mode b. Throughout time evolution, the quadrature operator $\hat{x}_b$ of mode b is conserved (i.e., generates a strong symmetry of the master equation). On the other hand, the unitary gates act as displacements on the conjugate quadrature $\hat{p}_b$. We can now understand system dynamics in terms of a stochastic unraveling via quantum trajectories: whenever a photon jump occurs in mode a, one would displace the quadrature $\hat{p}_b$ by a constant $-\theta/2$. Averaging over all stochastic trajectories thus gives rise to Eq. (G4).

We also note that although the equations of motion for the expectation value of b mode quadratures are relatively simple, the dynamics is still highly non-Gaussian. To see this explicitly, we first derive the equations of motion of higher-order moments of the cavity quadrature as

$$\frac{d\langle\hat{p}_b^2\rangle}{dt} = -\gamma\theta\langle\hat{p}_b\hat{a}^\dagger\hat{a}\rangle + \gamma\frac{\theta^2}{4}\langle\hat{a}^\dagger\hat{a}\rangle, \tag{G5}$$

$$\frac{d\langle\hat{p}_b^3\rangle}{dt} = -\gamma\frac{3\theta}{2}\langle\hat{p}_b^2\hat{a}^\dagger\hat{a}\rangle + \gamma\frac{3\theta^2}{4}\langle\hat{p}_b\hat{a}^\dagger\hat{a}\rangle - \gamma\frac{\theta^3}{8}\langle\hat{a}^\dagger\hat{a}\rangle, \tag{G6}$$

so that we obtain the equation of motion of the third-order cumulant $\langle \delta\hat{p}_b^3 \rangle$ as

$$\frac{d\langle \delta\hat{p}_b^3 \rangle}{dt} = -\gamma \frac{3\theta}{2} \langle \delta\hat{p}_b^2 \hat{a}^\dagger \hat{a} \rangle + \gamma \frac{3\theta}{2} \langle \delta\hat{p}_b^2 \rangle \langle \hat{a}^\dagger \hat{a} \rangle + \gamma \frac{3\theta^2}{4} \langle \delta\hat{p}_b \hat{a}^\dagger \hat{a} \rangle - \gamma \frac{\theta^3}{8} \langle \hat{a}^\dagger \hat{a} \rangle. \quad \text{(G7)}$$

If the system starts in a product state between two cavity modes, all but the last term on the right hand side of Eq. (G7) will vanish at initial time, leading to a nonzero third cumulant $\langle \delta\hat{p}_b^3(\delta t) \rangle$ at a positive infinitesimal time $\delta t > 0$ (as long as the cavity a has a nonzero photon number). Thus, the dynamics generated by Eq. (G1) is in general non-Gaussian.

[1] M. J. Bowick, N. Fakhri, M. C. Marchetti, and S. Ramaswamy, Symmetry, Thermodynamics, and Topology in Active Matter, Phys. Rev. X **12**, 010501 (2022).

[2] M. Fruchart, R. Hanai, P. B. Littlewood, and V. Vitelli, Non-reciprocal phase transitions, Nature **592**, 363 (2021).

[3] S. Shankar, A. Souslov, M. J. Bowick, M. C. Marchetti, and V. Vitelli, Topological active matter, Nat. Rev. Phys. **4**, 380 (2022).

[4] K. Stannigel, P. Rabl, and P. Zoller, Driven-dissipative preparation of entangled states in cascaded quantum-optical networks, New J. Phys. **14**, 063014 (2012).

[5] H. Pichler, T. Ramos, A. J. Daley, and P. Zoller, Quantum optics of chiral spin networks, Phys. Rev. A **91**, 042116 (2015).

[6] P. Lodahl, S. Mahmoodian, S. Stobbe, A. Rauschenbeutel, P. Schneeweiss, J. Volz, H. Pichler, and P. Zoller, Chiral quantum optics, Nature **541**, 473 (2017).

[7] K. Fang, Z. Yu, and S. Fan, Realizing effective magnetic field for photons by controlling the phase of dynamic modulation, Nat. Photon. **6**, 782 (2012).

[8] N. A. Estep, D. L. Sounas, J. Soric, and A. Alù, Magnetic-free non-reciprocity and isolation based on parametrically modulated coupled-resonator loops, Nat. Phys. **10**, 923 (2014).

[9] H. Nassar, B. Yousefzadeh, R. Fleury, M. Ruzzene, A. Alù, C. Daraio, A. N. Norris, G. Huang, and M. R. Haberman, Nonreciprocity in acoustic and elastic materials, Nat. Rev. Mater. **5**, 667 (2020).

[10] A. Kamal, J. Clarke, and M. H. Devoret, Noiseless non-reciprocity in a parametric active device, Nat. Phys. **7**, 311 (2011).

[11] B. Abdo, K. Sliwa, S. Shankar, M. Hatridge, L. Frunzio, R. Schoelkopf, and M. Devoret, Josephson Directional Amplifier for Quantum Measurement of Superconducting Circuits, Phys. Rev. Lett. **112**, 167701 (2014).

[12] K. M. Sliwa, M. Hatridge, A. Narla, S. Shankar, L. Frunzio, R. J. Schoelkopf, and M. H. Devoret, Reconfigurable Josephson Circulator/Directional Amplifier, Phys. Rev. X **5**, 041020 (2015).

[13] F. Ruesink, M.-A. Miri, A. Alù, and E. Verhagen, Nonreciprocity and magnetic-free isolation based on optomechanical interactions, Nat. Commun. **7**, 13662 (2016).

[14] F. Lecocq, L. Ranzani, G. A. Peterson, K. Cicak, R. W. Simmonds, J. D. Teufel, and J. Aumentado, Nonreciprocal Microwave Signal Processing with a Field-Programmable Josephson Amplifier, Phys. Rev. Appl. **7**, 024028 (2017).

[15] K. Fang, J. Luo, A. Metelmann, M. H. Matheny, F. Marquardt, A. A. Clerk, and O. Painter, Generalized non-reciprocity in an optomechanical circuit via synthetic magnetism and reservoir engineering, Nat. Phys. **13**, 465 (2017).

[16] G. A. Peterson, F. Lecocq, K. Cicak, R. W. Simmonds, J. Aumentado, and J. D. Teufel, Demonstration of Efficient Nonreciprocity in a Microwave Optomechanical Circuit, Phys. Rev. X **7**, 031001 (2017).

[17] N. R. Bernier, L. D. Tóth, A. Koottandavida, M. A. Ioannou, D. Malz, A. Nunnenkamp, A. K. Feofanov, and T. J. Kippenberg, Nonreciprocal reconfigurable microwave optomechanical circuit, Nat. Commun. **8**, 604 (2017).

[18] H. Xu, L. Jiang, A. A. Clerk, and J. G. E. Harris, Non-reciprocal control and cooling of phonon modes in an optomechanical system, Nature **568**, 65 (2019).

[19] D. Malz, L. D. Tóth, N. R. Bernier, A. K. Feofanov, T. J. Kippenberg, and A. Nunnenkamp, Quantum-Limited Directional Amplifiers with Optomechanics, Phys. Rev. Lett. **120**, 023601 (2018).

[20] C. W. Gardiner, Driving a Quantum System with the Output Field from Another Driven Quantum System, Phys. Rev. Lett. **70**, 2269 (1993).

[21] H. J. Carmichael, Quantum Trajectory Theory for Cascaded Open Systems, Phys. Rev. Lett. **70**, 2273 (1993).

[22] A. Metelmann and A. A. Clerk, Nonreciprocal Photon Transmission and Amplification via Reservoir Engineering, Phys. Rev. X **5**, 021025 (2015).

[23] A. Metelmann and A. A. Clerk, Nonreciprocal quantum interactions and devices via autonomous feedforward, Phys. Rev. A **95**, 013837 (2017).

[24] F. Verstraete, M. M. Wolf, and J. I. Cirac, Quantum computation and quantum-state engineering driven by dissipation, Nat. Phys. **5**, 633 (2009).

[25] P. Zanardi and L. Campos Venuti, Coherent Quantum Dynamics in Steady-State Manifolds of Strongly Dissipative Systems, Phys. Rev. Lett. **113**, 240406 (2014).

[26] V. V. Albert, B. Bradlyn, M. Fraas, and L. Jiang, Geometry and Response of Lindbladians, Phys. Rev. X **6**, 041031 (2016).

[27] C. Arenz and A. Metelmann, Emerging unitary evolutions in dissipatively coupled systems, Phys. Rev. A **101**, 022101 (2020).

[28] B. Misra and E. C. G. Sudarshan, The Zeno's paradox in quantum theory, J. Math. Phys. **18**, 756 (1977).

[29] P. Facchi and S. Pascazio, Quantum Zeno Subspaces, Phys. Rev. Lett. **89**, 080401 (2002).

[30] P. Facchi and S. Pascazio, Quantum Zeno dynamics: Mathematical and physical aspects, J. Phys. A **41**, 493001 (2008).

[31] D. Layden, E. Martín-Martínez, and A. Kempf, Universal scheme for indirect quantum control, Phys. Rev. A **93**, 040301(R) (2016).

[32] Q. Xu, G. Zheng, Y.-X. Wang, P. Zoller, A. A. Clerk, and L. Jiang, Autonomous quantum error correction and fault-tolerant quantum computation with squeezed cat qubits, (2022), arXiv preprint ArXiv:2210.13406.

[33] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2010).

[34] J. Watrous, *The Theory of Quantum Information* (Cambridge University Press, Cambridge, 2018).

[35] A. Y. Kitaev, Quantum computations: Algorithms and error correction, Russ. Math. Surv. **52**, 1191 (1997).

[36] This calculation amounts to computing the diamond norm between two unitary maps. An analytical derivation of the diamond norm with such forms can be found in, e.g., Sec. 3.3 of Ref. [34].

[37] H.-K. Lau and M. B. Plenio, Universal Quantum Computing with Arbitrary Continuous-Variable Encoding, Phys. Rev. Lett. **117**, 100501 (2016).

[38] B. Buča, J. Tindall, and D. Jaksch, Non-stationary coherent quantum many-body dynamics through dissipation, Nat. Commun. **10**, 1730 (2019).

[39] B. Buca, C. Booker, and D. Jaksch, Algebraic theory of quantum synchronization and limit cycles under dissipation, SciPost Phys. **12**, 97 (2022).

[40] M. Mirrahimi, Z. Leghtas, V. V. Albert, S. Touzard, R. J. Schoelkopf, L. Jiang, and M. H. Devoret, Dynamically protected cat-qubits: A new paradigm for universal quantum computation, New J. Phys. **16**, 045014 (2014).

[41] R. Lescanne, M. Villiers, T. Peronnin, A. Sarlette, M. Delbecq, B. Huard, T. Kontos, M. Mirrahimi, and Z. Leghtas, Exponential suppression of bit-flips in a qubit encoded in an oscillator, Nat. Phys. **16**, 509 (2020).

[42] A. Grimm, N. E. Frattini, S. Puri, S. O. Mundhada, S. Touzard, M. Mirrahimi, S. M. Girvin, S. Shankar, and M. H. Devoret, Stabilization and operation of a Kerr-cat qubit, Nature **584**, 205 (2020).

[43] J. M. Gertler, B. Baker, J. Li, S. Shirol, J. Koch, and C. Wang, Protecting a bosonic qubit with autonomous quantum error correction, Nature **590**, 243 (2021).

[44] J. Lebreuilly, K. Noh, C.-H. Wang, S. M. Girvin, and L. Jiang, Autonomous quantum error correction and quantum computation, arXiv preprint ArXiv:2103.05007.

[45] B. Kümmerer and H. Maassen, The essentially commutative dilations of dynamical semigroups on M_n, Commun. Math. Phys. **109**, 1 (1987).

[46] F. Fagnola, J. E. Gough, H. I. Nurdin, and L. Viola, Mathematical models of Markovian dephasing, J. Phys. A **52**, 385301 (2019).

[47] H. M. Wiseman, Quantum theory of continuous feedback, Phys. Rev. A **49**, 2133 (1994).

[48] Y. Castin, H. Wallis, and J. Dalibard, Limit of Doppler cooling, J. Opt. Soc. Am. B **6**, 2046 (1989).

[49] J. I. Cirac, R. Blatt, P. Zoller, and W. D. Phillips, Laser cooling of trapped ions in a standing wave, Phys. Rev. A **46**, 2668 (1992).

[50] D. Leibfried, R. Blatt, C. Monroe, and D. Wineland, Quantum dynamics of single trapped ions, Rev. Mod. Phys. **75**, 281 (2003).

[51] S. Kono, K. Koshino, Y. Tabuchi, A. Noguchi, and Y. Nakamura, Quantum non-demolition detection of an itinerant microwave photon, Nat. Phys. **14**, 546 (2018).

[52] J.-C. Besse, S. Gasparinetti, M. C. Collodo, T. Walter, P. Kurpiers, M. Pechal, C. Eichler, and A. Wallraff, Single-Shot Quantum Nondemolition Detection of Individual Itinerant Microwave Photons, Phys. Rev. X **8**, 021003 (2018).

[53] J.-C. Besse, S. Gasparinetti, M. C. Collodo, T. Walter, A. Remm, J. Krause, C. Eichler, and A. Wallraff, Parity Detection of Propagating Microwave Fields, Phys. Rev. X **10**, 011046 (2020).

[54] Y. Shi, Z. Yu, and S. Fan, Limitations of nonlinear optical isolators due to dynamic reciprocity, Nat. Photon. **9**, 388 (2015).

[55] D. L. Sounas and A. Alú, Nonreciprocity based on nonlinear resonances, IEEE Antennas Wirel. Propag. Lett. **17**, 1958 (2018).

[56] F. Fratini, E. Mascarenhas, L. Safari, J.-P. Poizat, D. Valente, A. Auffèves, D. Gerace, and M. F. Santos, Fabry-Perot Interferometer with Quantum Mirrors: Nonlinear Light Transport and Rectification, Phys. Rev. Lett. **113**, 243601 (2014).

[57] A. Rosario Hamann, C. Müller, M. Jerger, M. Zanner, J. Combes, M. Pletyukhov, M. Weides, T. M. Stace, and A. Fedorov, Nonreciprocity Realized with Quantum Nonlinearity, Phys. Rev. Lett. **121**, 123601 (2018).

[58] N. Nefedkin, M. Cotrufo, A. Krasnok, and A. Alú, Dark-state induced quantum nonreciprocity, Adv. Quantum Technol., 2100112 (2022).

[59] M. A. Nielsen, A simple formula for the average gate fidelity of a quantum dynamical operation, Phys. Lett. A **303**, 249 (2002).

[60] We use script letters with overhead check marks to denote matrices acting on the complex linear space $\mathbb{C}^N$.

[61] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, 2002).

[62] F. Elste, S. M. Girvin, and A. A. Clerk, Quantum Noise Interference and Backaction Cooling in Cavity Nanomechanics, Phys. Rev. Lett. **102**, 207209 (2009).

[63] A. Sawadsky, H. Kaufer, R. M. Nia, S. P. Tarabrin, F. Y. Khalili, K. Hammerer, and R. Schnabel, Observation of Generalized Optomechanical Coupling and Cooling on Cavity Resonance, Phys. Rev. Lett. **114**, 043601 (2015).

[64] H. Pichler, G. Zhu, A. Seif, P. Zoller, and M. Hafezi, Measurement Protocol for the Entanglement Spectrum of Cold Atoms, Phys. Rev. X **6**, 041033 (2016).

[65] Y. Y. Gao, B. J. Lester, K. S. Chou, L. Frunzio, M. H. Devoret, L. Jiang, S. M. Girvin, and R. J. Schoelkopf, Entanglement of bosonic modes through an engineered exchange interaction, Nature **566**, 509 (2019).

[66] J. Dalibard, Y. Castin, and K. Mølmer, Wave-Function Approach to Dissipative Processes in Quantum Optics, Phys. Rev. Lett. **68**, 580 (1992).

[67] S. Lieu, Y.-J. Liu, and A. V. Gorshkov, Candidate for a passively-protected quantum memory in two dimensions, arXiv preprint ArXiv:2205.09767.

[68] G. H. Fredrickson and H. C. Andersen, Kinetic Ising Model of the Glass Transition, Phys. Rev. Lett. **53**, 1244 (1984).

[69] E. Zohar, J. I. Cirac, and B. Reznik, Quantum simulations of lattice gauge theories using ultracold atoms in optical lattices, Rep. Prog. Phys. **79**, 014401 (2015).

[70] L. Barbiero, C. Schweizer, M. Aidelsburger, E. Demler, N. Goldman, and F. Grusdt, Coupling ultracold matter to dynamical gauge fields in optical lattices: from flux attachment to $\mathbb{Z}_2$ lattice gauge theories, Sci. Adv. **5**, eaav7444 (2019).

[71] V. Lienhard, P. Scholl, S. Weber, D. Barredo, S. de Léséleuc, R. Bai, N. Lang, M. Fleischhauer, H. P. Büchler, T. Lahaye, and A. Browaeys, Realization of a Density-Dependent Peierls Phase in a Synthetic, Spin-Orbit Coupled Rydberg System, Phys. Rev. X **10**, 021031 (2020).

[72] A. Celi, B. Vermersch, O. Viyuela, H. Pichler, M. D. Lukin, and P. Zoller, Emerging Two-Dimensional Gauge Theories in Rydberg Configurable Arrays, Phys. Rev. X **10**, 021057 (2020).

[73] N. Khaneja and S. J. Glaser, Cartan decomposition of SU(2n) and control of spin systems, Chem. Phys. **267**, 11 (2001).

[74] C. Gardiner and P. Zoller, *Quantum Noise: A Handbook of Markovian and Non-Markovian Quantum Stochastic Methods with Applications to Quantum Optics, Springer Series in Synergetics* (Springer, Berlin, 2004).