Dense Coding with Locality Restriction on Decoders: Quantum Encoders versus Superquantum Encoders

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We investigate practical dense coding by imposing locality restrictions on decoders and symmetry restrictions on encoders within the resource theory of asymmetry framework. In this task, the sender Alice and the helper Fred preshare an entangled state. Alice encodes a classical message into the quantum state using a symmetric preserving encoder and sends the encoded state to Bob through a noiseless quantum channel. The receiver Bob and helper Fred are limited to performing quantum measurements satisfying certain locality restrictions to decode the classical message. We are interested in the ultimate dense coding capacity under these constraints. Our contributions are summarized as follows. First, we derive both one-shot and asymptotic optimal achievable transmission rates of the dense coding task under different encoder and decoder combinations. Surprisingly, our results reveal that the transmission rate cannot be improved even when the decoder is relaxed from one-way local operations and classical communication (LOCC) to two-way LOCC, separable measurements, and partial transpose positive measurements of the bipartite system. Second, depending on the class of allowed decoders with certain locality restrictions, we relax the class of encoding operations to superquantum encoders in the general probability theory framework and derive dense coding transmission rates under this setting. For example, when the decoder is fixed to a separable measurement, theoretically, a positive operation is allowed as an encoding operation. Remarkably, even under this superquantum relaxation, the transmission rate still cannot be lifted. This fact highlights the universal validity of our analysis on practical dense coding beyond quantum theory.

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I. INTRODUCTION

Dense coding (also known as superdense coding) is a quantum communication protocol that communicates classical bits of information by transmitting quantum bits (qubits), assuming that the sender and receiver preshare an entangled resource [1–12]. In this protocol, the sender encodes classical messages into part of the preshared resource and sends it to the receiver via a noiseless quantum channel. The receiver decodes messages by performing measurements of the bipartite system. Dense coding has been accredited as a building block in quantum information theory. Subsequent studies investigated the dense coding protocol when the quantum channel or the preshared quantum state is noisy. However, two practical aspects of dense coding severely limit its usage, rendering dense coding impossible at a large scale.

One practical aspect concerns decoding operations in dense coding. Indeed, existing researches assume that the unique *receiver* holds the receiving system transmitted from the sender as well as the other entanglement system $\mathcal{H}_F$ called the *helper*. However, it may cause an experimental difficulty because this type of decoder requires a joint measurement across two systems: the receiver's system and the helper's system. In particular, if the receiver has only the receiving system and the helper is far from

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the receiver, this measurement requires a noiseless quantum communication channel, which is extremely difficult for experimental realization. Therefore, it is natural from a practical viewpoint to impose locality conditions on the available set of decoding measurements between the receiving system and the helper.

For example, suppose that the receiver has only the receiving system and can classically and freely communicate to the helper. They may adopt the available set of one-way local operations and classical communication (LOCC) measurements. When the receiver and the helper can classically and freely communicate with each other, they can adopt the set of two-way LOCC measurements. Since this class of measurements does not need quantum communication, it is experimentally easier. As larger classes of decoders, they can even adopt separable measurements and partial transpose positive (PPT) measurements. Though these two classes of measurements do not possess a clear operational interpretation yet, they have simple mathematical characterizations. Hence, they are helpful in proving the impossibility part. In this paper, we show that the class of one-way LOCC measurements can achieve the same performance as the class of measurements in the dense coding task. More specifically, the class of one-way LOCC measurements can extract sufficient benefit from shared entanglement without the use of joint measurements across two distinct parties, although one-way LOCC measurements are much more accessible than other complicated measurements. Since one-way LOCC measurements are more experimentally feasible, this result is an experimentally friendly demonstration of the advantage of entanglement assistance in dense coding protocols.

Indeed, enormous research has imposed locality conditions on quantum information processing tasks such as quantum state discrimination and quantum state verification [13–59]. However, few papers have addressed the locality issue that naturally emerges in dense coding [11], and this problem remains largely open.

Another practical aspect concerns encoding operations in dense coding. The sender commonly chooses unitary operations in the system $\mathcal{H}_A$ as encoding operations. However, it is not easy to implement arbitrary unitaries on the sender's system in practice. The time evolution in the quantum system can be characterized as unitary e^{itH} with Hamiltonian H. Practically, the one-parameter subgroup $\{e^{itH}\}$ can be easily implemented. When several types of Hamiltonian can be implemented, the subgroup generated by them can also be implemented. In this way, it is natural to restrict our encoding operations to a certain subgroup, under desirable symmetry constraints [60,61]. In particular, using this type of encoding operation, we demonstrate the practicality of the class with one-way LOCC measurements by a physically implementable example in Sec. V C.

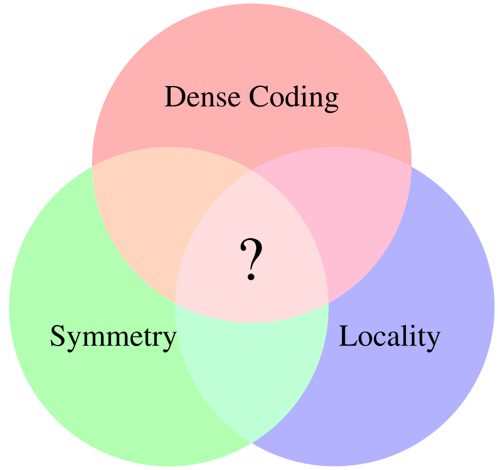

FIG. 1. The intersection of dense coding, symmetric encoders, and local decoders remains largely unknown. Better understanding this intersection would inspire more practical dense coding protocols.

In this paper, we study in depth the intersection of dense coding, symmetric encoders, and local decoders, wishing to explore the fundamental limits on symmetric encoders and local decoders and seek more practical dense coding protocols. See Fig. 1 for an illustration.

Recently, various studies addressed how quantum theory can be characterized in the context of general probability theory (GPT). Most of them discussed it in the context of state discrimination by extending the class of quantum measurements to a larger class of measurement in the framework of GPT [62–66], [66–80]. Specifically, Arai *et al.* [65] clarified that a superquantum measurement, i.e., a measurement in such an extended class, can distinguish two nonorthogonal states when our measurement belongs to the dual cone of the cone of separable operators. It was also shown that a similar phenomenon happens even when the cone of our measurements is very close to the cone of quantum theory [64,65]. This clarifies how quantum theory can be characterized in the framework of GPT. Also, Popescu and Rohrlich [67] showed the possibility of an unphysical state under the framework of GPT by introducing the PR box. In this paper, we go one step further to investigate dense coding with the framework of GPT. Though this line of research might not be meaningful from the practical perspective, it offers novel theoretical contribution to the foundation of quantum theory.

In fact, a positive map can generate an unphysical state from an entangled state when it is not a completely positive map. This is a reason why the set of positive maps is not considered as a class of quantum operations and it is described by the set of completely positive maps. Hence, a positive map can be considered as a superquantum operation when it is not a completely positive map. Therefore, it is a natural question whether such a superquantum operation enhances quantum information processing. In fact, the difference between positive maps and completely positive maps has not been studied in the context of quantum information processing. That is, such a study clarifies what property is essential for the analysis of quantum information processing in a broad setting, including quantum theory. From this aim, we address this question in the framework of GPT. In this paper, we focus on the information transmission problem as a typical model of quantum information processing. When a superquantum operation is allowed as an encoding operation in addition to a conventional quantum operation, the decoder needs to be restricted to a smaller class of measurements under the framework of GPT.

To consider the effect of an unphysical state generated by a superquantum operation, it is natural to consider the following situation; the sender and the receiver share an entangled state $|\Psi_{AF}\rangle$ and the encoding operation is limited to an operation on the sender's system $\mathcal{H}_A$, while a superquantum operation is allowed as an encoder. The decoder is restricted to a smaller class of measurements on the composite system under a suitable locality condition, which is chosen under the framework of GPT. Since this encoding scheme may generate an unphysical state, this problem setting clarifies the power of a superquantum encoder. In fact, several papers [64] studied state discrimination in the framework of GPT, but no study discussed the channel coding in the framework of GPT. In this sense, this analysis opens a new direction to the study of the foundation of quantum theory.

This paper is organized as follows. In Sec. II, we briefly explain the problem formulation of this paper. In Sec. III, we set the notation and prepare the basic knowledge used in the paper. In Sec. IV, we first formally define the abstract dense coding task. Then, we generalize the problem setting by considering various available sets of encoders constrained by resource theory of asymmetry and various available sets of decoders with locality conditions. Finally, we show that all these capacities are equal and derive a single-letter capacity formula. In Sec. V, we consider as examples various unitary groups of practical interest—the irreducible case, including the full unitary group; quantum coherence, including the one-generator case with a certain condition, e.g., a two-mode squeezed vacuum state; and Schur duality—to illustrate the dense coding power within different specialized resource theories of asymmetry. Finally, in Sec. VI we extend the obtained results to the case of a nonquantum preshared state within the framework of GPT. We defer all proofs to the appendices. In Appendix A, we summarize the lemmas and tools frequently used in the paper. In Appendix B, we prove an upper bound on the asymmetry of assistance and its regularized version. In Appendix C, we prove the weak and strong converse bounds on the (enhanced) dense coding capacities. In Appendix D, we prove our main result—the dense coding capacity theorem under locality conditions. We do so by first giving an one-shot characterization to the dense coding capacity with one-way LOCC decoders. This is done by showing an achievability bound in terms of the smooth Rényi entropy that is the most difficult part in this paper. Then, we derive the capacity formulas for the asymptotic dense coding capacities under the locality conditions. In Appendix E, we prove the dense coding theorem with local decoders even when superquantum encoding operations are allowed. In Appendix F, we prove the achievability (under certain conditions) and strong converse parts regarding the nonquantum preshared state extension.

II. BRIEF EXPLANATION OF THE FORMULATION

As explained in the Introduction, we restrict our encoding operations to a certain subgroup in the simplest setting. We assume that the encoding operation is given as a (projective) unitary representation U of a group G on $\mathcal{H}_A$. When the preshared entangled state is $|\Psi_{AF}\rangle$ and our encoding operation is restricted to these unitaries, our channel can be written as the classical-quantum (CQ) channel $g \mapsto U_g |\Psi_{AF}\rangle$. Since this CQ channel has a symmetric property for group G, we say that it is a *CQ-symmetric* channel. Recently, Korzekwa *et al.* [60] studied such a channel model in the context of resource theory of asymmetry without considering shared entanglement. In fact, resource theory of asymmetry is a topic to study physical resources for information processing, and has been actively studied by many researchers [12,81–91]. This method has also been used to measure the degree of noncommutativity in the context of quantum hypothesis testing [92], [93, Section 2.4].

The class of CQ-symmetric channels is a quantum generalization of a regular channel [94], which is a useful class of channels in classical information theory. This class of classical channels is often called generalized additive [95, Section V] or conditional additive [95, Section 4] and contains a class of additive channels as a subclass. Such a channel appears even in wireless communication by considering binary phase-shift keying modulations [96, Section 4.3]. Its most simple example is the binary symmetric channel. Hayashi [97, Section VII-A-2] studied its quantum extension with an additive group and discussed the capacity and the wire-tap capacity with the semantic security. Since this class has an excellent symmetric property, algebraic codes achieve the capacity [94,96–99]. Since an algebraic code has less calculation complexity in comparison with other types of code, this fact shows the usefulness of this class of classical channels. As the class of CQ-symmetric channels is a quantum version of classical channels, and the encoding operation class of group representation of a subgroup yields a CQ-symmetric channel, this encoding operation class is a natural class of encoders.

To consider superquantum encoders, we need to expand the above class of encoding operations because these unitaries are conventional quantum operations. To this end, we focus on the property of these unitaries. These unitary encoding operations have invariant states, which can be characterized by the average operation $\mathcal{G}$ of the given (projective) unitary representation. In the case of the full unitary and the discrete Weyl-Heisenberg groups, the average operation $\mathcal{G}$ maps all densities to the completely mixed state. When the group is composed of diagonal unitaries, the average operation $\mathcal{G}$ is the dephasing channel. Interestingly, the average operation $\mathcal{G}$ satisfies the property $\mathcal{G} \circ \mathcal{U}_g = \mathcal{U}_g \circ \mathcal{G} = \mathcal{G}$ for $g \in G$, where $\mathcal{U}_g(\rho) := U_g \rho U_g^\dagger$. Hence, as a larger class of encoding operations, we can consider the set of completely positive and trace-preserving (CPTP) maps $\{\mathcal{E}\}_\mathcal{E}$ that satisfies the symmetric preserving condition

$$\mathcal{G} \circ \mathcal{E} = \mathcal{E} \circ \mathcal{G} = \mathcal{G}. \tag{1}$$

Therefore, we assume that our encoders are restricted to the above class of CPTP maps as another problem setting.

From condition (1), we can define a class of trace-preserving positive maps (but not completely positive) as a larger class of encoders. Indeed, when we focus on a basis commutative with invariant states, the transpose operation satisfies condition (1). Hence, this class of encoders contains a typical superquantum operation. In addition, since our measurement class is restricted, this class is theoretically allowed as a class of encoding operations under the framework of GPT.

Suppose that our measurement class is smaller than the set of quantum measurements allowed in quantum theory. In that case, a larger class of states is allowed, i.e., a larger class of encoding operations is theoretically permitted in this framework. The reason is that the probability distribution of the measurement outcome is well defined in this relaxation. In other words, the non-negativity of the probability of the measurement outcome is guaranteed even under this relaxation. For example, when our measurement is restricted to a separable measurement, even when the encoding operation is relaxed to a positive map, the non-negativity of the probability of the measurement outcome is guaranteed, while the resultant state of the encoding is not necessarily positive definite. That is, the separability of our measurement guarantees the non-negativity of the probability of the measurement outcome. In this way, we can extend our encoding operation to such superquantum operations under condition (1) when a particular locality condition is imposed on our decoding measurement.

In this paper, we introduce **21** classes of dense coding codes by considering various classes of encoders and decoders. These classes are classified into three groups depending on the class of decoders. The first group comprises classes whose decoder has no support by $\mathcal{H}_F$. The second group comprises classes whose decoder is a global measurement. The remaining group comprises classes whose decoder has support from $\mathcal{H}_F$ and the locality condition on the bipartite system. As our main result, we show that every group class has the same capacity. That is, if a class belongs to the same group as another class, these two classes have the same capacity. Hence, when the available decoder is one of the bipartite decoders with locality conditions, even when the class of our encoders is extended to a larger class, e.g., trace-preserving positive maps with condition (1), the capacity cannot be improved.

TABLE I. Overview of the notation used throughout this paper.

Symbol	Definition
$\mathbb{C}, \mathbb{R}, \mathbb{R}^+$	Complex, real, and non-negative real numbers
$A, B, F, \ldots$	Quantum systems and the associated Hilbert spaces
$\mathscr{L}(\mathcal{H})$	Set of linear operators on system A
$\mathscr{P}(\mathcal{H})$	Set of positive semidefinite operators on system A
$\mathscr{D}(\mathcal{H})$	Set of density operators on system A
$\mathscr{C}(A \to B)$	Set of completely positive trace-preserving maps
$D(\rho\|\sigma)$	Quantum relative entropy of ρ and σ
$H(\rho_A), H(A)_\rho$	Von Neumann entropy of quantum state ρ_A
$\widetilde{D}_\alpha, \overline{D}_\alpha$	Sandwiched and Petz quantum Rényi divergences
$\mathscr{F}_G$	Set of symmetric states with respect to a group G
$\mathcal{G}$	Twirling operation of a group G
$A_G(\rho)$	Asymmetry of assistance of quantum state ρ
$A_G^\infty(\rho)$	Regularized asymmetry of assistance of ρ
$\mathfrak{E}_c$	Set of available encoders under constraint c
$\mathfrak{D}_c$	Set of available decoders under constraint c
$C^\varepsilon_{\mathfrak{E},\mathfrak{D}}(\Psi_{AF})$	One-shot ε-dense coding capacity of Ψ_{AF}
$C_{\mathfrak{E},\mathfrak{D}}(\Psi_{AF})$	Dense coding capacity of a pure state Ψ_{AF}
$C^\dagger_{\mathfrak{E},\mathfrak{D}}(\Psi_{AF})$	Strong converse dense coding capacity of Ψ_{AF}

III. PRELIMINARIES

In this section, we first set the notation. Then we review the mathematical tool of quantum entropies and group representation. Finally, we review the resource theory of asymmetry and introduce two new asymmetry measures: asymmetry of assistance and its regularized version. We summarize the notation used throughout the paper in Table I for reference.

A. Notation

For a finite-dimensional Hilbert space $\mathcal{H}$, we denote by $\mathscr{L}(\mathcal{H})$ and $\mathscr{P}(\mathcal{H})$ the linear and positive semidefinite operators on $\mathcal{H}$. Quantum states are in the set $\mathscr{D}(\mathcal{H}) :=$

$\{\rho \in \mathscr{P}(\mathcal{H}) | \operatorname{Tr}\rho = 1\}$ and we also define the set of subnormalized quantum states $\mathscr{D}_\bullet(\mathcal{H}) := \{\rho \in \mathscr{P}(\mathcal{H}) | 0 < \operatorname{Tr}\rho \leq 1\}$. For two operators $M, N \in \mathscr{L}(\mathcal{H})$, we say that $M \geq N$ if and only if $M - N \in \mathscr{P}(\mathcal{H})$. On the other hand, we denote by $\{M \geq N\}$ the projector onto the space spanned by the eigenvectors of $M - N$ with non-negative eigenvalues. The identity matrix is denoted as $\mathbb{1}$, and the maximally mixed state is denoted as π. Multipartite quantum systems are described by tensor product spaces. We use Latin capital letters to denote the different systems and subscripts to indicate what subspace an operator acts. For example, if M_{AB} is an operator on $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ then $M_A = \operatorname{Tr}_B M_{AB}$ is defined as its marginal on system A. Systems with the same letter are assumed to be isomorphic: $A' \cong A$. By convention, we use letters in front of alphabet letters such as A and B to represent quantum systems and letters after alphabet letters such as X and Y to represent classical systems. We say that ρ_{XA} is a classical-quantum state if it is of the form $\rho_{XA} = \sum_x p_X(x)|x\rangle\langle x|_X \otimes \rho_A^x$, where p_X is a probability distribution, $\{|x\rangle\}_x$ is an orthonormal basis of $\mathcal{H}_X$, and $\{\rho_A^x \in \mathscr{D}(\mathcal{H}_A)\}_x$. A linear map $\mathcal{N} : \mathscr{L}(\mathcal{H}_A) \to \mathscr{L}(\mathcal{H}_B)$ maps operators in system A to operators in system B. If $\mathcal{N}_{A\to B}(M_A) \in \mathscr{P}(\mathcal{H}_B)$ whenever $M_A \in \mathscr{P}(\mathcal{H}_A)$, $\mathcal{N}_{A\to B}$ is positive. Let id_A denote the identity map acting on system A. If the map $\mathrm{id}_R \otimes \mathcal{N}_{A\to B}$ is positive for every reference system R, $\mathcal{N}_{A\to B}$ is completely positive (CP). If $\operatorname{Tr}[\mathcal{N}_{A\to B}(M_A)] = \operatorname{Tr} M_A$ for all operators $M_A \in \mathscr{L}(\mathcal{H}_A)$, $\mathcal{N}_{A\to B}$ is trace preserving (TP). If $\mathcal{N}_{A\to B}$ is completely positive and trace preserving (CPTP), we say that it is a quantum channel or quantum operation. We denote by $\mathscr{C}(A \to B)$ the set of quantum channels from A to B. A positive operator-valued measure (POVM) is a set $\{\Lambda_m\}$ of operators satisfying, for all m, $\Lambda_m \geq 0$ and $\sum_m \Lambda_m = \mathbb{1}$. Denote by $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{R}^+$ the complex, real, and non-negative real numbers, respectively.

B. Quantum entropies

Let $\rho \in \mathscr{D}(\mathcal{H})$ and $\sigma \in \mathscr{P}(\mathcal{H})$ such that the support of ρ is contained in the support of σ. The quantum relative entropy is defined as

$$D(\rho\|\sigma) := \operatorname{Tr}[\rho(\log\rho - \log\sigma)], \tag{2}$$

where logarithms are in base 2 throughout this paper. The Shannon entropy of a probability distribution p_X is defined as $H(p_X) := -\sum_x p_X(x)\log p_X(x)$. The von Neumann entropy of ρ is defined as $H(\rho) := -\operatorname{Tr}\rho\log\rho$. Let $\rho_{AB} \in \mathscr{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ be a bipartite quantum state. The quantum mutual information and conditional entropy of ρ_{AB} are respectively defined as

$$I(A{:}B)_\rho := D(\rho_{AB}\|\rho_A \otimes \rho_B), \tag{3}$$

$$H(A|B)_\rho := -D(\rho_{AB}\|\mathbb{1}_A \otimes \rho_B). \tag{4}$$

Trivializing system B, $H(A|B)_\rho$ yields an alternative definition of the von Neumann entropy as $H(A)_\rho$. In this paper, we use such notation interchangeably. The quantum information variance is defined as [100]

$$V(\rho\|\sigma) := \operatorname{Tr}[\rho(\log\rho - \log\sigma)^2] - D(\rho\|\sigma)^2. \tag{5}$$

The varentropy (aka variance and information variance) of ρ is defined as

$$V(\rho) := V(\rho\|\mathbb{1}) = \operatorname{Tr}[\rho(\log\rho)^2] - H(\rho)^2. \tag{6}$$

Let $\alpha \in (0,1) \cup (1,\infty)$; the one-parameter Petz quantum Rényi divergence is defined as [101] (we refer the interested reader to Ref. [102, Chapter 4] and Ref. [103, Secs. 3.1 and 5.4] for a comprehensive study of $\overline{D}_\alpha$):

$$\overline{D}_\alpha(\rho\|\sigma) := \frac{1}{\alpha - 1}\log\operatorname{Tr}[\rho^\alpha\sigma^{1-\alpha}]. \tag{7}$$

As another version, we focus on the one-parameter sandwiched quantum Rényi divergence $\widetilde{D}_\alpha$ defined as [104–107]

$$\widetilde{D}_\alpha(\rho\|\sigma) := \frac{1}{\alpha - 1}\log\operatorname{Tr}[(\sigma^{(1-\alpha)/2\alpha}\rho\sigma^{(1-\alpha)/2\alpha})^\alpha]. \tag{8}$$

Interestingly, both quantum Rényi divergence recovers the quantum relative entropy by taking the limit $\alpha \to 1$:

$$\lim_{\alpha\to 1}\overline{D}_\alpha(\rho\|\sigma) = \lim_{\alpha\to 1}\widetilde{D}_\alpha(\rho\|\sigma) = D(\rho\|\sigma). \tag{9}$$

The quantum Rényi entropy of ρ is defined as

$$\overline{H}_\alpha(\rho) := -\overline{D}_\alpha(\rho\|\mathbb{1}) = \frac{1}{1-\alpha}\log\operatorname{Tr}\rho^\alpha. \tag{10}$$

Equation (9) yields $\lim_{\alpha\to 1}\overline{H}_\alpha(\rho) = H(\rho)$. The Petz conditional Rényi entropy of a bipartite state ρ_{AB} is defined as

$$\overline{H}_\alpha(A|B)_\rho := -\overline{D}_\alpha(\rho_{AB}\|\mathbb{1}_A \otimes \rho_B). \tag{11}$$

C. Group representation

As a preparation to explain the resource theory of asymmetry, we summarize the basic fact in group representation. Let $\mathcal{H}$ be a Hilbert space and G be a group. For an element $g \in G$, a unitary operator U_g is given. The map $U : g \mapsto U_g$ is called a unitary representation of G on $\mathcal{H}$ when

$$U_g U_{g'} = U_{gg'} \tag{12}$$

for $g, g' \in G$ [93]. In addition, the map U is called a projective unitary representation of G on $\mathcal{H}$ when there exists

$\theta(g, g')$ for $g, g' \in G$ such that [93]

$$U_g U_{g'} = e^{i\theta(g,g')} U_{gg'}. \tag{13}$$

In particular, the above type of a projective unitary representation is called a projective unitary representation associated with $\{\theta(g, g')\}_{g,g' \in G}$.

A (projective) unitary representation U of G on $\mathcal{H}$ is called irreducible when there is no subspace $\mathcal{K} \subset \mathcal{H}$ such that $U_g \mathcal{K} \subset \mathcal{K}$ for $g \in G$ and $\{0\} \neq \mathcal{K}$. Two (projective) unitary representations U_1, U_2 of G on $\mathcal{H}_1, \mathcal{H}_2$ are called equivalent when there exists a unitary V from $\mathcal{H}_1$ to $\mathcal{H}_2$ such that $VU_{1,g}V^\dagger = U_{2,g}$ for $g \in G$. A (projective) unitary representation U of G on $\mathcal{H}$ is called completely reducible when it is a direct sum representation of (projective) irreducible unitary representations. It is known that any unitary representation U of G on $\mathcal{H}$ is completely reducible when G is a compact group [108, Lemma 2.3].

Let $\hat{G}$ be the set of indices that identifies an irreducible unitary representation of G. That is, given an element $k \in \hat{G}$, we have an irreducible unitary representation U_k of G on $\mathcal{H}_k$. Then, any unitary representation U of a compact G on $\mathcal{H}$ is equivalent to a unitary representation on the representation space

$$\bigoplus_{k \in \hat{G}} \mathcal{H}_k \otimes \mathbb{C}^{n_k}, \tag{14}$$

where n_k is called the multiplicity of the irreducible unitary representation U_k. However, when the multiplicity n_k is not one, the multiplicity causes a technical difficulty. Hence, we introduce the following assumption. We explain the importance of this assumption in Sec. IV G, after we present the main results. In fact, as explained in Sec. V, several typical examples satisfy this assumption.

Assumption 1 (Multiplicity-free condition). *We say that a unitary representation U of a group G on $\mathcal{H}$ is multiplicity-free when there exists a subset $S \subset \hat{G}$ such that the unitary representation U on $\mathcal{H}$ is equivalent to a unitary representation on the representation space*

$$\bigoplus_{k \in S} \mathcal{H}_k. \tag{15}$$

We assume that our unitary representation satisfies the multiplicity-free condition throughout this manuscript.

The above discussion can be extended to projective unitary representations. Let $\hat{G}[\{\theta(g, g')\}_{g,g' \in G}]$ be the set of indices to identify an irreducible projective unitary representation of G associated with $\{\theta(g, g')\}_{g,g' \in G}$. Then, Eq. (14) is generalized as follows. Any unitary representation U of a compact G on $\mathcal{H}$ associated with $\{\theta(g, g')\}_{g,g' \in G}$ is equivalent to a unitary representation on the representation space

$$\bigoplus_{k \in \hat{G}[\{\theta(g,g')\}_{g,g' \in G}]} \mathcal{H}_k \otimes \mathbb{C}^{n_k}, \tag{16}$$

where n_k is called the multiplicity of the irreducible unitary representation U_k. Hence, we define the property "multiplicity-free" for a projective unitary representation U in the same way.

A state $\sigma \in \mathscr{D}(\mathcal{H})$ is *symmetric* with respect to G if it holds that

$$\mathcal{U}_g(\sigma) \equiv U_g \sigma U_g^\dagger = \sigma \quad \text{for all } g \in G. \tag{17}$$

That is, the symmetric states are invariant under G. Throughout this paper, we assume that the group G is fixed and omit the explicit reference to this group. The set of symmetric states is denoted as $\mathscr{F}_G$ and will be treated as *free states* in the resource theory of asymmetry. Conversely, a state $\rho \in \mathscr{D}(\mathcal{H})$ is *asymmetric*, or resourceful, if there exists some $g \in G$ such that $U_g \rho U_g^\dagger \neq \rho$. When the group G is a finite group, the *G-twirling operation* $\mathcal{G}$ over G is defined as

$$\mathcal{G}(\rho) := \frac{1}{|G|} \sum_g U_g \rho U_g^\dagger. \tag{18}$$

When the group G is a compact group, the above definition can be generalized as

$$\mathcal{G}(\rho) := \int_G U_g \rho U_g^\dagger \nu(dg), \tag{19}$$

where ν is the Haar measure. Operation $\mathcal{G}$ maps all states in $\mathscr{D}(\mathcal{H})$ to symmetric states, i.e.,

$$\mathcal{G}(\rho) \in \mathscr{F}_G \quad \text{for all } \rho \in \mathscr{D}(\mathcal{H}). \tag{20}$$

What is more, $\mathcal{G}$ is symmetry preserving in the sense that it maps any symmetric state to itself: $\mathcal{G}(\sigma) = \sigma$ for any symmetric state $\sigma \in \mathscr{F}_G$. One can interpret $\mathcal{G}$ as a resource destroying map [109] in the sense that it leaves resource-free states unchanged but erases the resource stored in all resourceful states.

Lemma 1. *Let G is a compact group. A (projective) unitary representation U is multiplicity-free if and only if*

$$\mathcal{G}(\rho)\mathcal{G}(\sigma) = \mathcal{G}(\sigma)\mathcal{G}(\rho) \tag{21}$$

for two states ρ, σ on $\mathcal{H}$.

Proof. Assume that a (projective) unitary representation U is multiplicity-free, as required in Eq. (15). Let Π_k be the projection to the space $\mathcal{H}_k$. Then, $\mathcal{G}(\rho) = \bigoplus_{k\in S} \mathrm{Tr}(\Pi_k\rho)\Pi_k/\mathrm{Tr}\,\Pi_k$, which implies Eq. (21). Conversely, assume that a (projective) unitary representation U is not multiplicity-free. Then, a state ρ is written as $\bigoplus_k \rho_k$, where ρ_k is a positive semidefinite operator on $\mathcal{H}_k \otimes \mathbb{C}^{n_k}$. Then, $\mathcal{G}(\rho) = \bigoplus_{k\in S} \Pi_k/\mathrm{Tr}\,\Pi_k \otimes (\mathrm{Tr}_{\mathcal{H}_k} \rho_k)$. For $n_k > 1$, $\mathrm{Tr}_{\mathcal{H}_k} \rho_k$ and $\mathrm{Tr}_{\mathcal{H}_k} \sigma_k$ are not commutative with each other in general. Hence, we obtain the desired statement. ■

The above definition generalizes naturally to a tensor product system $\mathcal{H}^{\otimes n}$ composed of n copies of $\mathcal{H}$. The group in $\mathcal{H}^{\otimes n}$ is $G^{\times n}$ and we adopt the notation $\mathbf{g} \equiv g_1 \cdots g_n$ and $U_{\mathbf{g}} \equiv U_{g_1} \otimes \cdots \otimes U_{g_n}$ for each $g_i \in G$. Symmetric states are defined to be those satisfying $U_{\mathbf{g}}\sigma U_{\mathbf{g}} = \sigma$ for all $U_{\mathbf{g}} \in G^{\times n}$. Correspondingly, the twirling operation in $\mathcal{H}^{\otimes n}$ is $\mathcal{G}^{\otimes n}$.

D. Resource theory of asymmetry

Asymmetry of quantum states plays an important role not only in the development of modern physics but also in quantum information processing tasks [81]. In this section, we briefly summarize the *resource theory of asymmetry* [12,81–90], which is a special case of a general formalism named the quantum resource theory [110]. We remark that the resource theory of asymmetry is an *abstract* resource theory that encapsulates many nice properties of commonly studied resource theories in the literature [111,112].

The relative entropy of asymmetry [83] is a commonly used measure that quantifies the degree of asymmetry of quantum states and is defined as

$$R_G(\rho) := \min_{\sigma\in\mathscr{F}_G} D(\rho\|\sigma). \tag{22}$$

It turns out that the twirled state $\mathcal{G}(\rho)$ achieves the minimum in Eq. (22) and yields a simple expression for the relative entropy of asymmetry in terms of the von Neumann entropy [83, Proposition 2], i.e.,

$$R_G(\rho) = D(\rho\|\mathcal{G}(\rho)) = H(\mathcal{G}(\rho)) - H(\rho). \tag{23}$$

This quantity is the minimum relative entropy between ρ and the set of invariant states and is called the relative entropy of G-frameness [91, Proposition 2]. Using this quantity, Hayashi [93, Theorem 2.9] showed the Pythagorean theorem for quantum relative entropy in the sense of group invariant space. When G is a commutative group generated by the logarithm of another density matrix σ, Hiai and Petz [92] considered this quantity as the degree of noncommutativity between ρ and σ. They showed that the regularization of this quantity goes to zero when ρ and σ are given as n-tensor products. This fact was used in the proof of quantum Stein's lemma for quantum hypothesis testing [92]. Gour *et al.* [91, Corollary 11] extended this fact to the case when the group is the n-tensor product of any group representation. In this sense, quantity (23) takes various roles in quantum information.

Inspired by the entanglement of assistance [113] in the resource theory of entanglement [114] and the coherence of assistance [115] in the resource theory of coherence [112], we introduce here the *asymmetry of assistance* of a quantum state ρ as

$$\begin{aligned} A_G(\rho) &:= \max_{\rho=\sum_x p_X(x)|\psi_x\rangle\langle\psi_x|} \sum_x p_X(x) D(|\psi_x\rangle\langle\psi_x| \\ &\quad \times \|\mathcal{G}(|\psi_x\rangle\langle\psi_x|)) \\ &= \max_{\rho=\sum_x p_X(x)|\psi_x\rangle\langle\psi_x|} \sum_x p_X(x) H(\mathcal{G}(|\psi_x\rangle\langle\psi_x|)), \end{aligned} \tag{24}$$

where $\psi_x \equiv |\psi_x\rangle\langle\psi_x|$, the maximum ranges over all possible pure state decompositions of ρ, and the second equality follows from Eq. (23). Correspondingly, the *regularized asymmetry of assistance* of ρ is defined as

$$A_G^\infty(\rho) := \limsup_{n\to\infty} \frac{1}{n} A_G(\rho^{\otimes n}). \tag{25}$$

In the following proposition, we show that both A_G and A_G^∞ are upper bounded by the quantum entropy of the twirled state, and thus that the regularization is well defined. The proof can be found in Appendix B.

Proposition 2. *Let $\rho \in \mathscr{D}(\mathcal{H})$ be a quantum state. It holds that*

$$A_G(\rho) \le A_G^\infty(\rho) \le H(\mathcal{G}(\rho)). \tag{26}$$

When G is the commutative group composed of diagonal unitaries, and ρ is a pure state, Chitambar *et al.* [115] discussed $A_G(\rho)$ and $A_G^\infty(\rho)$. Regarding Eq. (26), they showed that the equality in the second inequality for the general case [115, Theorem 4] and the equality in the first inequality for the qubit case [115, Theorem 5]. In addition, they showed that there exists a pure state ρ that does not satisfy the equality in the first inequality when the dimension of $\mathcal{H}$ is not smaller than 4. They stated that the equality in the first inequality is an open problem in the qutrit case.

IV. DENSE CODING CAPACITIES

A. The general dense coding framework

We first describe the most general dense coding framework. Let the preshared entangled state between Alice and Fred be $|\Psi\rangle_{AF}$, the set of available encoders by Alice be $\mathfrak{E}$, and the set of available decoders by Bob and Fred be

$\mathfrak{D}$. The abstract dense coding protocol can be described as follows. Alice randomly samples a message m from the message alphabet $\mathcal{M}$ and then applies an encoding channel $\mathcal{E}^m_{A\to A} \in \mathfrak{E}$ to the resourceful state Ψ_{AF}. This leads to the classical-quantum state

$$\frac{1}{|\mathcal{M}|}\sum_m |m\rangle\langle m|_M \otimes \mathcal{E}^m_{A\to A}(|\Psi\rangle\langle\Psi|_{AF}). \tag{27}$$

After encoding, Alice sends the encoded state to Bob via a noiseless quantum channel $\mathrm{id}_{A\to B}$ where $A \cong B$. After receiving the quantum state, Bob and Fred perform a joint measurement $\mathcal{D}_{BF\to\widehat{M}} \equiv \{\Gamma^{\widehat{m}}_{BF}\}_{\widehat{m}} \in \mathfrak{D}$ to infer the encoded message m. See Fig. 2 for an illustration of the dense coding protocol. The decoding operation results in the classical-classical quantum state

$$\sum_{m,\widehat{m}} q_{\widehat{M}M}(\widehat{m}|m)|m\rangle\langle m|_M \otimes |\widehat{m}\rangle\langle\widehat{m}|_{\widehat{M}}, \tag{28}$$

where the conditional distribution $q_{\widehat{M}M}$ is defined as

$$q_{\widehat{M}M}(\widehat{m}|m) := \mathrm{Tr}[\Gamma^{\widehat{m}}_{BF}\mathcal{E}^m_{A\to A}(|\Psi\rangle\langle\Psi|_{AF})]. \tag{29}$$

We call $\mathcal{C} \equiv (\{\mathcal{E}^m\}_m, \mathcal{D}) \in (\mathfrak{E}, \mathfrak{D})$ a *dense coding code* for the resourceful quantum state Ψ_{AF} under the available encoder-decoder pair $(\mathfrak{E}, \mathfrak{D})$ with cardinality $|\mathcal{C}| \equiv |\mathcal{M}|$. We quantify the performance of $\mathcal{C}$ by computing the *decoding error*

$$e(\mathcal{C}) := 1 - \frac{1}{|\mathcal{M}|}\sum_m q_{\widehat{M}M}(m|m), \tag{30}$$

and use $s(\mathcal{C}) := 1 - e(\mathcal{C})$ to denote the success probability of decoding. In general, smaller decoding error implies better code. However, to achieve small $e(\mathcal{C})$, one has to encode with small size $|\mathcal{C}|$. This motivates us to define the *dense coding rate* that quantitatively measures the communication capacity of the code:

$$r(\mathcal{C}) := \log|\mathcal{C}|. \tag{31}$$

Fix $\varepsilon \in [0, 1)$. The one-shot ε-dense coding capacity of Ψ_{AF} under available encoder-decoder pair $(\mathfrak{E}, \mathfrak{D})$ is defined to be the maximum bits of messages that can be transmitted such that the decoding error is upper bounded by the error threshold ε.

Definition 3 (One-shot ε-dense coding capacity)**.** *Let $|\Psi\rangle_{AF}$ be a bipartite pure quantum state and $\varepsilon \in [0, 1)$. The one-shot ε-dense coding capacity of Ψ_{AF} under available encoder-decoder pair $(\mathfrak{E}, \mathfrak{D})$ is defined as*

$$C^{\varepsilon}_{\mathfrak{E},\mathfrak{D}}(\Psi_{AF}) := \sup_{\mathcal{C}\in(\mathfrak{E},\mathfrak{D})}\{r(\mathcal{C}) \mid e(\mathcal{C}) \le \varepsilon\}. \tag{32}$$

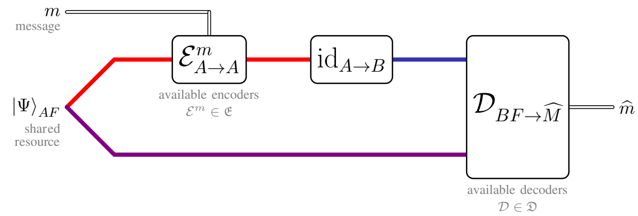

FIG. 2. A dense coding protocol for the shared resourceful quantum state Ψ_{AF} under the available encoder-decoder pair $(\mathfrak{E}, \mathfrak{D})$. In this protocol, Alice possesses system A (red line), Bob possesses system B (blue line), and Fred possesses system F (purple line).

The dense coding capacity of Ψ_{AF} under available encoder-decoder pair $(\mathfrak{E}, \mathfrak{D})$ is then defined to be the one-shot ε-dense coding capacity of $\Psi^{\otimes n}_{AF}$ by taking the limits $n \to \infty$ and $\varepsilon \to 0$. This capacity quantifies the ultimate number of bits that can be reliably transmitted per copy of Ψ_{AF} in the asymptotic regime, under available encoder-decoder pair $(\mathfrak{E}, \mathfrak{D})$.

Definition 4 (Dense coding capacity)**.** *Let $|\Psi\rangle_{AF}$ be a bipartite pure quantum state. The dense coding capacity of Ψ_{AF} under available encoder-decoder pair $(\mathfrak{E}, \mathfrak{D})$ is defined as*

$$C_{\mathfrak{E},\mathfrak{D}}(\Psi_{AF}) := \inf_{\varepsilon>0}\limsup_{n\to\infty}\frac{1}{n}C^{\varepsilon}_{\mathfrak{E},\mathfrak{D}}(\Psi^{\otimes n}_{AF}). \tag{33}$$

Analogously, the strong converse dense coding capacity of Ψ_{AF} is defined to be the one-shot ε-dense coding capacity of $\Psi^{\otimes n}_{AF}$ by taking the limit $n \to \infty$ and satisfying the constraint that $\varepsilon < 1$. This capacity quantifies the extent to which we can sacrifice the decoding error to achieve a larger dense coding rate in the asymptotic regime, under available encoder-decoder pair $(\mathfrak{E}, \mathfrak{D})$.

Definition 5 (Strong converse dense coding capacity)**.** *Let $|\Psi\rangle_{AF}$ be a bipartite pure quantum state. The strong converse dense coding capacity of Ψ_{AF} under available encoder-decoder pair $(\mathfrak{E}, \mathfrak{D})$ is defined as*

$$C^{\dagger}_{\mathfrak{E},\mathfrak{D}}(\Psi_{AF}) := \sup_{\varepsilon<1}\limsup_{n\to\infty}\frac{1}{n}C^{\varepsilon}_{\mathfrak{E},\mathfrak{D}}(\Psi^{\otimes n}_{AF}). \tag{34}$$

In the following sections, we introduce various available classes of encoders $\mathfrak{E}$ and decoders $\mathfrak{D}$ within the resource theory of asymmetry framework.

B. Quantum and superquantum encoders

Practically, it is not easy to implement arbitrary quantum operations for an encoder. Hence, it is natural to restrict Alice's encoding operation to a particular class of operations. For example, when a Hamiltonian H is fixed, the unitary operation e^{itH} can be easily implemented. Noting that the set $\{e^{itH}\}_t$ forms a group representation, this example can be generalized as follows. Given a group G and its unitary (projective) representation U_g, we assume the following available set of Alice's encoding operations:

$$\mathfrak{E}_g := \{\mathcal{U}_g \mid g \in G\} \tag{35}$$

with $\mathcal{U}_g$ defined in Eq. (17). We note that $\mathfrak{E}_g$ remains as our first and smallest set of encoders.

However, in general, the dephasing operation can be easily experimentally implemented, yet it is not included in $\mathfrak{E}_g$. This motivates us to enlarge $\mathfrak{E}_g$ to encapsulate physically implementable quantum operations. Along this line, Korzekwa *et al.* [60] proposed enlarging $\mathfrak{E}_g$ to the following available set of encoding operations that *commute* with $\mathcal{G}$:

$$\mathfrak{E}_{\rm cp} := \{\mathcal{E} \in \mathscr{C}(A \to A)_{\rm cp} \mid \mathcal{E} \circ \mathcal{G} = \mathcal{G} = \mathcal{G} \circ \mathcal{E}\} \tag{36}$$

with $\mathscr{C}(A \to A)_{\rm cp}$ the set of CPTP maps from A to A. This class is larger than $\mathfrak{E}_g$. When the Hamiltonian is H and the set $\mathfrak{E}_g$ is given as the set $\{e^{itH}\}_t$, the dephasing operation is contained in $\mathfrak{E}_{\rm cp}$, matching our requirement.

To investigate the power of *superquantum* encoders, we introduce the following class of encoding operations:

$$\mathfrak{E}_p := \{\mathcal{E} \in \mathscr{C}(A \to A)_p \mid \mathcal{E} \circ \mathcal{G} = \mathcal{G} = \mathcal{G} \circ \mathcal{E}\} \tag{37}$$

with $\mathscr{C}(A \to A)_p$ the set of trace-preserving and positive maps from A to A. From the perspective of resource theory of asymmetry, each quantum operation $\mathcal{E} \in \mathfrak{E}_p$ can encode information (both classical and quantum) into some degrees of freedom of resourceful states that can be completely destroyed by $\mathcal{G}$. Since $\mathscr{C}(A \to A)_p$ is strictly larger than $\mathscr{C}(A \to A)_{\rm cp}$, there is a possibility of enhancing the dense coding capacity by using the set of encoders $\mathfrak{E}_p$ over $\mathfrak{E}_{\rm cp}$.

In addition, as an intermediate set between $\mathscr{C}(A \to A)_p$ and $\mathscr{C}(A \to A)_{\rm cp}$, we introduce the set $\mathscr{C}(A \to A)_{\rm ppt}$ as

$$\begin{aligned}\mathscr{C}(A \to A)_{\rm ppt} := \{\mathcal{E} \in \mathscr{C}(A \to A)_p \mid \mathrm{Tr} \\ \times [(\mathrm{id}_F \otimes \mathcal{E})(|\Phi\rangle\langle\Phi|)\rho] \geq 0\}\end{aligned} \tag{38}$$

for any PPT state ρ on $\mathcal{H}_{AF}$, where Φ is a maximally entangled state on the bipartite system $\mathcal{H}_{AF}$. As another virtual setting, we may also consider the set of encoders

$$\mathfrak{E}_{\rm ppt} := \{\mathcal{E} \in \mathscr{C}(A \to A)_{\rm ppt} \mid \mathcal{E} \circ \mathcal{G} = \mathcal{G} = \mathcal{G} \circ \mathcal{E}\}. \tag{39}$$

By studying the above two classes of encoders, we can clarify whether superquantum encoders can enhance classical information transmission with a preshared resourceful quantum state. We can show the following inclusion hierarchy for the four classes of encoders defined above:

$$\mathfrak{E}_g \subset \mathfrak{E}_{\rm cp} \subset \mathfrak{E}_{\rm ppt} \subset \mathfrak{E}_p. \tag{40}$$

Indeed, the set of positive maps from the two-dimensional system to itself is generated by the CPTP maps and the transpose operation [116]. This fact shows the equality $\mathfrak{E}_{\rm ppt} = \mathfrak{E}_p$. However, Horodecki, [117] and Skowronek [118] showed that the set of positive maps from the three-dimensional system to itself requires infinitely many generators. This fact indicates the possibility of enhancing the dense coding capacity by using the set of encoders $\mathfrak{E}_p$ over $\mathfrak{E}_{\rm ppt}$ when $\dim \mathcal{H}_A \geq 3$.

C. Decoders under locality conditions

Many studies investigated dense coding protocols under the assumptions that entanglement is preshared and arbitrary encoding and decoding operations are allowed. However, even when the sender and the receiver share an entangled state, it is not easy to implement a general joint measurement across two quantum systems—the message receiver $\mathcal{H}_B$ and the entanglement receiver $\mathcal{H}_F$. Hence, it is natural to impose *locality conditions* for the decoders.

As a typical case, we can consider the *one-way LOCC decoders* $\mathfrak{D}_{\rightarrow}$ where the classical communication flows from the entanglement receiver $\mathcal{H}_F$ to the message receiver $\mathcal{H}_B$. In this case, the entanglement receiver Fred first measures the shared state at hand and then shares the information with the message receiver Bob via classical communication. Conditioned on the information, Bob decodes the message using local decoders. Aiming to improve the dense coding capacity, we also introduce the *two-way LOCC decoders* $\mathfrak{D}_{\leftrightarrow}$, where the two-way LOCC operations can be realized by combinations of local operations and classical communications between the two systems. We remark that (one-way) LOCC decoders are the most natural set of quantum operations in distributed quantum information processing.

Motivated by the resource theory of quantum entanglement [114], we may further enlarge the set of available decoder POVMs with respect to the bipartite system B:F to improve the communication rate by considering *separable decoders* (also known as separable measurements) $\mathfrak{D}_{\rm sep}$ and *PPT decoders* (also known as PPT measurements) $\mathfrak{D}_{\rm ppt}$. Intuitively, a joint measurement is a separable (PPT) measurement if all of its POVM elements can be implemented by separable operations (PPT operations). Although the separable decoders $\mathfrak{D}_{\rm sep}$ and the PPT decoders $\mathfrak{D}_{\rm ppt}$ are theoretical objects, they are useful in proving the converse part in coding theorems.

Finally, we consider two special cases covering the commonly studied dense coding tasks. First, we investigate the *local decoders* $\mathfrak{D}_{\not\leftrightarrow}$ in which Bob decodes the message encoded by Alice without help from Fred (the entanglement receiver), which we call nonassisted decoding. Second, we investigate the *global decoders* $\mathfrak{D}_{\rm glb}$ in which Bob and Fred work together to decode the message encoded by Alice. We do not impose any locality condition on the joint measurements they can carry out. Note that global decoders are used in the seminal dense coding protocol originally proposed by Bennett and Wiesner [1].

We conclude the following inclusion hierarchy for the six classes of decoders defined above:

$$\mathfrak{D}_{\not\leftrightarrow} \subset \mathfrak{D}_{\rightarrow} \subset \mathfrak{D}_{\leftrightarrow} \subset \mathfrak{D}_{\rm sep} \subset \mathfrak{D}_{\rm ppt} \subset \mathfrak{D}_{\rm glb}. \tag{41}$$

D. Landscape of dense coding capacities

In the above two sections, we have proposed four classes of available encoders—$\mathfrak{E}_g, \mathfrak{E}_{\rm cp}, \mathfrak{E}_{\rm ppt}, \mathfrak{E}_p$—and six classes of available decoders—$\mathfrak{D}_{\not\leftrightarrow}, \mathfrak{D}_{\rightarrow}, \mathfrak{D}_{\leftrightarrow}, \mathfrak{D}_{\rm sep}, \mathfrak{D}_{\rm ppt}, \mathfrak{D}_{\rm glb}$. However, it is not the case that the arbitrary encoder-decoder pair chosen from the available sets can form a valid code for the resourceful quantum state Ψ_{AF}. More precisely, consider the encoder-decoder pair $(\mathfrak{E}, \mathfrak{D})$ where $\mathfrak{E} \in \{\mathfrak{E}_g, \mathfrak{E}_{\rm cp}, \mathfrak{E}_{\rm ppt}, \mathfrak{E}_p\}$ and $\mathfrak{D} \in \{\mathfrak{D}_{\not\leftrightarrow}, \mathfrak{D}_{\rightarrow}, \mathfrak{D}_{\leftrightarrow}, \mathfrak{D}_{\rm sep}, \mathfrak{D}_{\rm ppt}, \mathfrak{D}_{\rm glb}\}$. The encoding operations $\{\mathcal{E}^m\}_m$ chosen from $\mathfrak{E}$ by Alice and the decoder POVM $\{\Gamma^m\}_m$ chosen from $\mathfrak{D}$ by Bob and Fred yield the conditional values $q_{\widehat{M}M}$ defined in Eq. (29). To guarantee that $q_{\widehat{M}M}$ is a conditional distribution [and the corresponding code $\mathcal{C} = (\{\mathcal{E}^m\}_m, \{\Gamma^m\}_m)$ is a valid dense coding code], it must hold that

$$q_{\widehat{M}M}(\widehat{m}|m) = \operatorname{Tr}[\Gamma^{\widehat{m}}_{BF}\mathcal{E}^m_{A\to A}(|\Psi\rangle\langle\Psi|_{AF})] \geq 0 \tag{42}$$

for arbitrary $m, \widehat{m} \in \mathcal{M}$. This physical constraint rules out the possible combinations $(\mathfrak{E}_{\rm ppt}, \mathfrak{D}_{\rm glb}), (\mathfrak{E}_p, \mathfrak{D}_{\rm ppt}), (\mathfrak{E}_p, \mathfrak{D}_{\rm glb})$, since these encoder-decoder pairs may lead to negative values. Conversely, all other possible pairs $(\mathfrak{E}, \mathfrak{D})$ (21 pairs in total) are valid encoder-decoder pairs for the dense coding protocol. For reference, we outline the landscape of investigated dense coding capacities in Table II.

Remark 1: In the extreme case where the measurement outcomes on $\mathcal{H}_F$ are completely *ignored*, i.e., only local decoders $\mathfrak{D}_{\not\leftrightarrow}$ are available, we recover the communication capacities of Ψ_A previously investigated in Ref. [60].

We summarize in the following inclusion relations for the dense coding capacities defined above. Let $|\Psi\rangle_{AF}$ be a bipartite pure quantum state. It holds that

$$C_{\mathfrak{E},\mathfrak{D}}(\Psi_{AF}) \leq C^{\dagger}_{\mathfrak{E},\mathfrak{D}}(\Psi_{AF}) \tag{43}$$

for $\mathfrak{E} \in \{\mathfrak{E}_g, \mathfrak{E}_{\rm cp}, \mathfrak{E}_{\rm ppt}, \mathfrak{E}_p\}$ and $\mathfrak{D} \in \{\mathfrak{D}_{\not\leftrightarrow}, \mathfrak{D}_{\rightarrow}, \mathfrak{D}_{\leftrightarrow}, \mathfrak{D}_{\rm sep}, \mathfrak{D}_{\rm ppt}, \mathfrak{D}_{\rm glb}\}$, except for the unphysical pairs $(\mathfrak{E}_{\rm ppt}, \mathfrak{D}_{\rm glb}), (\mathfrak{E}_p, \mathfrak{D}_{\rm ppt}), (\mathfrak{E}_p, \mathfrak{D}_{\rm glb})$. The relation hierarchy for the capacities $C_{\mathfrak{E},\mathfrak{D}}(\Psi_{AF})$ is illustrated in Fig. 3. Also, we have the same relation hierarchy for the strong converse capacities $C^{\dagger}_{\mathfrak{E},\mathfrak{D}}(\Psi_{AF})$.

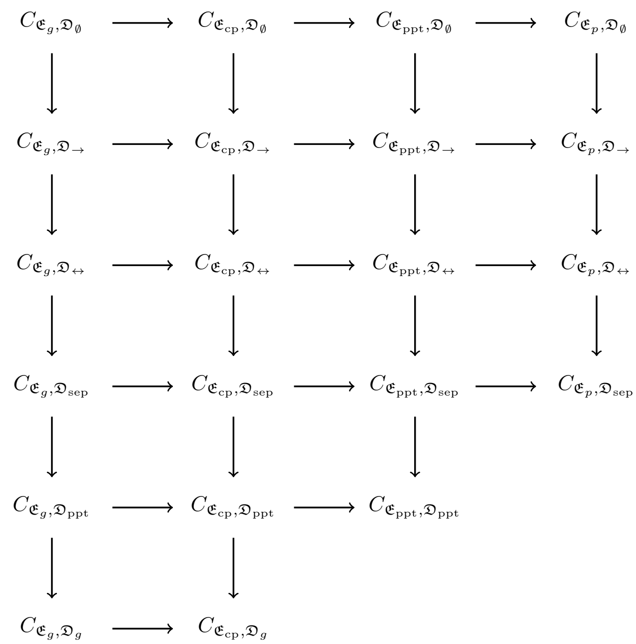

FIG. 3. The relation hierarchy for the dense coding capacities $C_{\mathfrak{E},\mathfrak{D}}(\Psi_{AF})$ investigated in this paper, where $x \to y$ means that $x \leq y$ in the hierarchy.

E. Enhanced version with one-way LOCC

In the above dense coding framework, if we fix the available decoders to $\mathfrak{D}_{\rightarrow}$, i.e., the one-way LOCC decoders, this specific setting has an equivalent description called the *environment-assisted classical communication via quantum resources*, originally motivated by the intensively studied environment assistance framework [113,115,119–131]. The detailed dense coding procedure using one-way LOCC decoders is illustrated in Fig. 4.

Inspired by Fig. 4, we propose here a hypothetical and *enhanced* dense coding framework with one-way LOCC decoders, in which both Alice and Bob have access to Fred's outcome, as illustrated in Fig. 5. This hypothetical setting yields upper bounds on the standard dense coding with one-way LOCC decoders. The dense coding power of Ψ_{AF} is enhanced compared to the setting depicted in Fig. 4

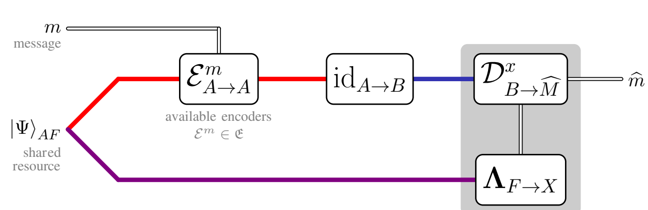

FIG. 4. A dense coding protocol for the shared resourceful quantum state Ψ_{AF} under one-way LOCC decoders (shaded area). In this protocol, Alice possesses system A (red line), Bob possesses system B (blue line), and Fred possesses system F (purple line).

TABLE II. Landscape of the dense coding capacities investigated in this paper. We are able to show that the capacities in bold are actually equal and derive single-letter capacity formulas for all these capacities under Assumption 1.

		Encoder $\mathfrak{E}$			
		Quantum encoder		Superquantum encoder	
Decoder $\mathfrak{D}$		$\mathfrak{E}_g$	$\mathfrak{E}_{\rm cp}$	$\mathfrak{E}_{\rm ppt}$	$\mathfrak{E}_p$
Local	$\mathfrak{D}_{\leftrightarrow\!\!\!\!/}$	$C_{\mathfrak{E}_g,\mathfrak{D}_{\leftrightarrow\!\!\!\!/}}(\Psi_{AF})$	$C_{\mathfrak{E}_{\rm cp},\mathfrak{D}_{\leftrightarrow\!\!\!\!/}}(\Psi_{AF})$	$C_{\mathfrak{E}_{\rm ppt},\mathfrak{D}_{\leftrightarrow\!\!\!\!/}}(\Psi_{AF})$	$C_{\mathfrak{E}_p,\mathfrak{D}_{\leftrightarrow\!\!\!\!/}}(\Psi_{AF})$
One-way LOCC	$\mathfrak{D}_{\rightarrow}$	$C_{\mathfrak{E}_g,\mathfrak{D}_{\rightarrow}}(\Psi_{AF})$	$C_{\mathfrak{E}_{\rm cp},\mathfrak{D}_{\rightarrow}}(\Psi_{AF})$	$C_{\mathfrak{E}_{\rm ppt},\mathfrak{D}_{\rightarrow}}(\Psi_{AF})$	$C_{\mathfrak{E}_p,\mathfrak{D}_{\rightarrow}}(\Psi_{AF})$
LOCC	$\mathfrak{D}_{\leftrightarrow}$	$C_{\mathfrak{E}_g,\mathfrak{D}_{\leftrightarrow}}(\Psi_{AF})$	$C_{\mathfrak{E}_{\rm cp},\mathfrak{D}_{\leftrightarrow}}(\Psi_{AF})$	$C_{\mathfrak{E}_{\rm ppt},\mathfrak{D}_{\leftrightarrow}}(\Psi_{AF})$	$C_{\mathfrak{E}_p,\mathfrak{D}_{\leftrightarrow}}(\Psi_{AF})$
Separable	$\mathfrak{D}_{\rm sep}$	$C_{\mathfrak{E}_g,\mathfrak{D}_{\rm sep}}(\Psi_{AF})$	$C_{\mathfrak{E}_{\rm cp},\mathfrak{D}_{\rm sep}}(\Psi_{AF})$	$C_{\mathfrak{E}_{\rm ppt},\mathfrak{D}_{\rm sep}}(\Psi_{AF})$	$C_{\mathfrak{E}_p,\mathfrak{D}_{\rm sep}}(\Psi_{AF})$
PPT	$\mathfrak{D}_{\rm ppt}$	$C_{\mathfrak{E}_g,\mathfrak{D}_{\rm ppt}}(\Psi_{AF})$	$C_{\mathfrak{E}_{\rm cp},\mathfrak{D}_{\rm ppt}}(\Psi_{AF})$	$C_{\mathfrak{E}_{\rm ppt},\mathfrak{D}_{\rm ppt}}(\Psi_{AF})$	✗
Global	$\mathfrak{D}_{\rm glb}$	$C_{\mathfrak{E}_g,\mathfrak{D}_{\rm glb}}(\Psi_{AF})$	$C_{\mathfrak{E}_{\rm cp},\mathfrak{D}_{\rm glb}}(\Psi_{AF})$	✗	✗

since Alice possesses additional information (from Fred). Following Definitions 3, 4, and 5, we can define analogously corresponding enhanced dense coding capacities introduced in Fig. 5 as

$$\widetilde{C}^{\varepsilon}_{\mathfrak{E},\mathfrak{D}_{\rightarrow}}(\Psi_{AF}),\quad \widetilde{C}_{\mathfrak{E},\mathfrak{D}_{\rightarrow}}(\Psi_{AF}),\quad \widetilde{C}^{\dagger}_{\mathfrak{E},\mathfrak{D}_{\rightarrow}}(\Psi_{AF}), \tag{44}$$

respectively, where $\mathfrak{E}\in\{\mathfrak{E}_g,\mathfrak{E}_{\rm cp},\mathfrak{E}_{\rm ppt},\mathfrak{E}_p\}$. Throughout this paper, we use the letter $\widetilde{C}$ with a tilde to represent the enhanced dense coding capacity. We conclude the following weak and strong converse bounds on the (enhanced) dense coding capacities. See Appendix C for the proof.

Theorem 1. *Let $|\Psi\rangle_{AF}$ be a bipartite pure quantum state and $\varepsilon\in[0,1)$. It holds that*

$$C^{\varepsilon}_{\mathfrak{E},\mathfrak{D}_{\rightarrow}}(\Psi_{AF})\le \widetilde{C}^{\varepsilon}_{\mathfrak{E},\mathfrak{D}_{\rightarrow}}(\Psi_{AF}), \tag{45a}$$

$$C_{\mathfrak{E},\mathfrak{D}_{\rightarrow}}(\Psi_{AF})\le \widetilde{C}_{\mathfrak{E},\mathfrak{D}_{\rightarrow}}(\Psi_{AF})\le A^{\infty}_{G}(\Psi_{A}), \tag{45b}$$

$$C^{\dagger}_{\mathfrak{E},\mathfrak{D}_{\rightarrow}}(\Psi_{AF})\le \widetilde{C}^{\dagger}_{\mathfrak{E},\mathfrak{D}_{\rightarrow}}(\Psi_{AF})\le H(\mathcal{G}(\Psi_{A})), \tag{45c}$$

where $\mathfrak{E}\in\{\mathfrak{E}_g,\mathfrak{E}_{cp},\mathfrak{E}_{ppt},\mathfrak{E}_p\}$. In the above relations, A^{∞}_{G} is defined in Eq. (25), $\Psi_A=\operatorname{Tr}_F\Psi_{AF}$, and the operation $\mathcal{G}$ is defined in Eqs. (18) and (19).

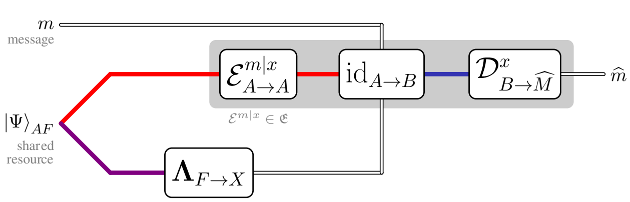

FIG. 5. An enhanced dense coding protocol for the shared resourceful quantum state Ψ_{AF} under one-way LOCC decoders. In this protocol, Alice possesses system A (red line), Bob possesses system B (blue line), and Fred possesses system F (purple line).

F. Main results

1. Dense coding capacities under locality conditions

Our main result concerns the dense coding capacities under various locality conditions—$\mathfrak{D}_{\rightarrow}$, $\mathfrak{D}_{\leftrightarrow}$, $\mathfrak{D}_{\rm sep}$, $\mathfrak{D}_{\rm ppt}$. In a word, we show that all these capacities are equal and derive a single-letter capacity formula. Before stating the result, we outline some notation first. We assume that the (projective) unitary representation U on $\mathcal{H}_A$ is multiplicity-free (cf. Assumption 1). The Hilbert space $\mathcal{H}_A$ is decomposed as $\bigoplus_{k\in\mathcal{K}}\mathcal{H}_k$. Hence, any pure state Ψ_{AF} on the bipartite system $\mathcal{H}_A\otimes\mathcal{H}_F$ can be written as

$$\Psi_{AF}=\sum_{k\in\mathcal{K}}\sqrt{P_K(k)}\Psi_{AF,k}, \tag{46}$$

where $\{P_K(k)\}_k$ is a probability distribution and each $\Psi_{AF,k}$ is a pure state on the bipartite system $\mathcal{H}_k\otimes\mathcal{H}_F$. The twirled state on the bipartite system $\mathcal{H}_A\otimes\mathcal{H}_F$ is given as

$$\xi_{AF}:=(\mathcal{G}_A\otimes\mathrm{id}_F)(\Psi_{AF})=\sum_{k\in\mathcal{K}}P_K(k)\pi_k\otimes\rho_{F,k}, \tag{47}$$

where $\rho_{F,k}:=\operatorname{Tr}_A\Psi_{AF,k}$ and π_k is the maximally mixed state on $\mathcal{H}_k$. The multiplicity-free condition guarantees the relation

$$\begin{aligned} H(\xi_{AF}) &= H((\mathcal{G}_A\otimes\mathrm{id}_F)(\Psi_{AF}))\\ &= H\left(\sum_k P_K(k)\pi_k\right)+\sum_k P_K(k)H(\rho_{F,k})\\ &= H(A)_\xi+H(F|K)_\xi. \end{aligned}\tag{48}$$

What is more,

$$\begin{aligned} H(A)_\xi &= H\left(\sum_k P_K(k)\pi_k\right)\\ &=\sum_{k\in\mathcal{K}}P_K(k)\log d_k+H(P_K), \end{aligned}\tag{49}$$

where d_k is the dimension of the irreducible subspace $\mathcal{H}_k$.

Our main result is summarized as follows, and the proof can be found in Appendix D.

Theorem 2 (Dense coding capacity under locality conditions)**.** *Let* $|\Psi\rangle_{AF}$ *be a bipartite pure quantum state. It holds under Assumption 1 (the multiplicity-free condition) that*

$$C_{\mathfrak{E},\mathfrak{D}}(\Psi_{AF}) = C^{\dagger}_{\mathfrak{E},\mathfrak{D}}(\Psi_{AF}) = A^{\infty}_{G}(\Psi_A) = H(\mathcal{G}(\Psi_A)) = H(A)_{\xi} \tag{50}$$

for $\mathfrak{E} \in \{\mathfrak{E}_g, \mathfrak{E}_{cp}, \mathfrak{E}_{ppt}, \mathfrak{E}_p\}$ *and* $\mathfrak{D} \in (\mathfrak{D}_{\rightarrow}, \mathfrak{D}_{\leftrightarrow}, \mathfrak{D}_{sep}, \mathfrak{D}_{ppt})$ *except for* $(\mathfrak{E}_p, \mathfrak{D}_{ppt})$*. In the above relations,* A^{∞}_{G} *is defined in Eq. (25),* $\Psi_A = \mathrm{Tr}_F\,\Psi_{AF}$*, and the operation* $\mathcal{G}$ *is defined in Eqs. (18) and (19). In addition, since* ξ_A *is defined in Eq. (47), the entropy* $H(A)_{\xi}$ *can be written as Eq. (49). What is more, the strong converse bound holds without Assumption 1.*

We highlight the importance of Theorem 2 as follows.

1. It reveals the fact that, even when we enlarge the available encoder-decoder pair up to $(\mathfrak{E}_p, \mathfrak{D}_{\mathrm{sep}})$ or $(\mathfrak{E}_{\mathrm{ppt}}, \mathfrak{D}_{\mathrm{ppt}})$, we cannot improve the dense coding capacity compared to minimal encoder-decoder pair $(\mathfrak{E}_g, \mathfrak{D}_{\rightarrow})$, where the available encoders are the unitary encoding operations and the available decoders are the one-way LOCC measurements.
2. It shows that the dense coding capacities under locality conditions all satisfy the desirable *strong converse property*. That is, for arbitrary dense coding code $\mathcal{C} \in (\mathfrak{E}, \mathfrak{D})$, where $\mathfrak{E} \in \{\mathfrak{E}_g, \mathfrak{E}_{\mathrm{cp}}, \mathfrak{E}_{\mathrm{ppt}}, \mathfrak{E}_p\}$ and $\mathfrak{D} \in (\mathfrak{D}_{\rightarrow}, \mathfrak{D}_{\leftrightarrow}, \mathfrak{D}_{\mathrm{sep}}, \mathfrak{D}_{\mathrm{ppt}})$ except for $(\mathfrak{E}_p, \mathfrak{D}_{\mathrm{ppt}})$, the decoding error necessarily converges to one in the asymptotic limit whenever the coding rate exceeds the optimal rate $H(\mathcal{G}(\Psi_A))$. We thus conclude that $H(\mathcal{G}(\Psi_A))$ is a very sharp dividing line between the coding rates that are achievable and those that are not, ruling out the possibility of error-rate trade-off in this dense coding task.
3. It establishes an interesting equivalence among three different quantities at first glance: the operationally defined dense coding capacity $C_{\mathfrak{E},\mathfrak{D}}(\Psi_{AF})$ (33), the mathematically defined regularized asymmetry of assistance $A^{\infty}_{G}(\Psi_A)$ (25), and the quantum entropy of the twirled quantum state $H(\mathcal{G}(\Psi_A))$. In this way, we provide the asymmetry measure A^{∞}_{G} with an operational meaning in terms of dense coding tasks.

Moreover, as a direct result of Theorems 1 and 2, we conclude that even if Alice has access to the measurement outcomes sent by Fred (cf. Fig. 5 for the enhanced dense coding framework), the dense coding capability cannot be improved when compared to the standard dense coding framework, where only Bob can access the measurement outcomes sent by Fred, when one-way LOCC decoders are available. This result, in some sense, indicates that Alice can choose the encoding operations completely independent of the encoded state and yields an "universal encoding" strategy under the same notation as Theorem 2.

Corollary 6. *Let* $|\Psi\rangle_{AF}$ *be a bipartite pure quantum state. It holds that*

$$\widetilde{C}_{\mathfrak{E},\mathfrak{D}_{\rightarrow}}(\Psi_{AF}) = \widetilde{C}^{\dagger}_{\mathfrak{E},\mathfrak{D}_{\rightarrow}}(\Psi_{AF}) = A^{\infty}_{G}(\Psi_A) = H(\mathcal{G}(\Psi_A)) = H(A)_{\xi}, \tag{51}$$

where $\mathfrak{E} \in \{\mathfrak{E}_g, \mathfrak{E}_{cp}, \mathfrak{E}_{ppt}, \mathfrak{E}_p\}$*.*

2. Dense coding capacities with local decoders

When only the local decoders $\mathfrak{D}_{\leftrightarrow}$ are available, we can derive the following coding theorem under the same notation as Theorem 2. Note that our results on local decoders recover as a special case Theorem 3 of Ref. [60], in which the $\mathfrak{E} = \mathfrak{E}_{\mathrm{cp}}$ case was considered. See Appendix E for the proof.

Theorem 3 (Dense coding capacity with local decoders)**.** *Let* $|\Psi\rangle_{AF}$ *be a bipartite pure quantum state. It holds under Assumption 1 (the multiplicity-free condition) that*

$$\begin{aligned} C_{\mathfrak{E},\mathfrak{D}_{\leftrightarrow}}(\Psi_{AF}) &= C^{\dagger}_{\mathfrak{E},\mathfrak{D}_{\leftrightarrow}}(\Psi_{AF}) \\ &= H(\mathcal{G}(\Psi_A)) - H(\Psi_A) \\ &= D\left(\Psi_A \| \mathcal{G}(\Psi_A)\right) \end{aligned} \tag{52}$$

for $\mathfrak{E} \in \{\mathfrak{E}_g, \mathfrak{E}_{cp}, \mathfrak{E}_{ppt}, \mathfrak{E}_p\}$*.*

3. Dense coding capacities with global decoders

When the global decoders $\mathfrak{D}_{\mathrm{glb}}$ are available, we actually identify an variant of the well-known dense coding task [1–9,11,12] in which the available encoders in system A are constrained by the twirling operation $\mathcal{G}_A$. Note that the following result has previously been discovered in Ref. [60, Theorem 3].

Proposition 7 (Dense coding capacity with global decoders)**.** *Let* $|\Psi\rangle_{AF}$ *be a bipartite pure quantum state. It holds that*

$$\begin{aligned} C_{\mathfrak{E}_g,\mathfrak{D}_{glb}}(\Psi_{AF}) &= C_{\mathfrak{E}_{cp},\mathfrak{D}_{glb}}(\Psi_{AF}) \\ &= H((\mathcal{G}_A \otimes \mathrm{id}_F)(\Psi_{AF})). \end{aligned} \tag{53}$$

4. Comparisons of the dense coding capacities

Now we compare the dense coding capacities investigated above to witness the power of different decoders.

Using Eq. (48), Theorem 3 can be rewritten in terms of ξ_{AF} as

$$\begin{aligned} C_{\mathfrak{E},\mathfrak{D}_{\nleftrightarrow}}(\Psi_{AF}) &= C^{\dagger}_{\mathfrak{E},\mathfrak{D}_{\nleftrightarrow}}(\Psi_{AF}) \\ &= H(\mathcal{G}(\Psi_A)) - H(\Psi_A) \\ &= H(A)_\xi - H(F)_\xi, \end{aligned} \tag{54}$$

and Proposition 7 can be rewritten in terms of ξ_{AF} as

$$\begin{aligned} C_{\mathfrak{E}_g,\mathfrak{D}_{\rm glb}}(\Psi_{AF}) &= C_{\mathfrak{E}_{\rm cp},\mathfrak{D}_{\rm glb}}(\Psi_{AF}) \\ &= H((\mathcal{G}_A \otimes {\rm id}_F)(\Psi_{AF})) \\ &= H(A)_\xi + H(F|K)_\xi. \end{aligned} \tag{55}$$

Comparing Eqs. (50) and (54), we can see that the dense coding capacity under locality conditions can be interpreted as the sum of the amount of asymmetry $H(A)_\xi - H(F)_\xi$ reserved in the quantum state Ψ_A and the amount of assistance $H(F)_\xi$ from the nonlocal decoders $\mathfrak{D} \in \{\mathfrak{D}_\rightarrow, \mathfrak{D}_\leftrightarrow, \mathfrak{D}_{\rm sep}, \mathfrak{D}_{\rm ppt}\}$. On the other hand, Eqs. (50) and (55) together imply that $H(F|K)_\xi$ can be viewed as the merit of global decoders compared to the nonlocal decoders $\mathfrak{D} \in \{\mathfrak{D}_\rightarrow, \mathfrak{D}_\leftrightarrow, \mathfrak{D}_{\rm sep}, \mathfrak{D}_{\rm ppt}\}$.

G. Role of Assumption 1

Assumption 1 plays a crucial role in deriving the above results. To see the importance of Assumption 1, we discuss a fundamental lemma related to group representation theory, which will be used in our proof of Theorem 2.

We consider a pure state $|\psi\rangle$ in a general representation space $\mathcal{H}$ of a representation U_g of a group G. We consider an irreducible decomposition $\mathcal{H} = \bigoplus_{k\in\mathcal{K}} \mathcal{H}_k$ of $\mathcal{H}$, where $\mathcal{H}_k$ is an irreducible representation space. We denote the projection to $\mathcal{H}_k$ by Π_k. Then, we have the following lemma.

Lemma 8. *When Assumption 1 holds, we have*

$$\mathcal{G}(|\psi\rangle\langle\psi|) = \sum_{k\in\mathcal{K}} \mathcal{G}(\Pi_k|\psi\rangle\langle\psi|\Pi_k). \tag{56}$$

Proof. Assumption 1 guarantees the uniqueness of the decomposition $\mathcal{H} = \bigoplus_{k\in\mathcal{K}} \mathcal{H}_k$. Since $U_g\mathcal{G}(|\psi\rangle\langle\psi|) = \mathcal{G}(|\psi\rangle\langle\psi|)U_g$, Schur's lemma guarantees that $\mathcal{G}(|\psi\rangle\langle\psi|)$ is written as $\sum_k c_k\Pi_k$ with coefficients c_k. Hence, the cross term with respect to the irreducible decomposition $\mathcal{H} = \bigoplus_{k\in\mathcal{K}} \mathcal{H}_k$ vanishes after application of $\mathcal{G}$. Therefore, we have

$$\mathcal{G}(|\psi\rangle\langle\psi|) = \sum_{k\in\mathcal{K}} \Pi_k\mathcal{G}(|\psi\rangle\langle\psi|)\Pi_k = \sum_{k\in\mathcal{K}} \mathcal{G}(\Pi_k|\psi\rangle\langle\psi|\Pi_k). \tag{57}$$

This completes the proof. ■

Indeed, when Assumption 1 does not hold, relation (56) does not hold in general. In this case, after application of $\mathcal{G}$, there are terms across several irreducible spaces $\mathcal{H}_{k_1}, \ldots, \mathcal{H}_{k_l}$, which are equivalent irreducible spaces. Lemma 8 plays an essential role in our direct part. Specifically, this lemma is used to prove Lemma 15, which is presented in step 4 of the proof of Theorem 6 given in Appendix D. On the other hand, Theorem 6 concludes the one-shot direct part of Theorem 2.

V. EXAMPLES

In this section, we compute the dense coding capacities $C_{\mathfrak{E}_g,\mathfrak{D}_{\nleftrightarrow}}(\Psi_{AF})$, $C_{\mathfrak{E}_g,\mathfrak{D}_\rightarrow}(\Psi_{AF})$, $C_{\mathfrak{E}_g,\mathfrak{D}_{\rm glb}}(\Psi_{AF})$ for specialized resource theories of asymmetry of practical interest. Note that, by Theorems 2 and 3 and Proposition 7, it suffices to evaluate these three capacities.

A. Dense coding power of purity

First, we consider the case when the (projective) unitary representation U is *irreducible* on $\mathcal{H}_A$. For example, G can be the group of unitary matrices on $\mathcal{H}_A$. Also, when G is the discrete Weyl-Heisenberg group on $\mathcal{H}_A$, the corresponding U forms an irreducible projective unitary representation. In this case, the twirling operation $\mathcal{G}$ becomes the completely depolarizing channel such that $\mathcal{G}(\rho_A) = \mathbb{1}_A/d_A$ for all $\rho_A \in \mathscr{D}(\mathcal{H}_A)$, where d_A is the dimension of system A. Correspondingly, the induced resource theory is known as the resource theory of purity [111,132,133]. For this resource theory, we have

$$C_{\mathfrak{E}_g,\mathfrak{D}_{\nleftrightarrow}}(\Psi_{AF}) = \log d_A - H(\Psi_A), \tag{58a}$$

$$C_{\mathfrak{E}_g,\mathfrak{D}_\rightarrow}(\Psi_{AF}) = \log d_A, \tag{58b}$$

$$C_{\mathfrak{E}_g,\mathfrak{D}_{\rm glb}}(\Psi_{AF}) = \log d_A + H(\Psi_A), \tag{58c}$$

where the first equality follows from a special case of Theorem 3, the second equality follows from Theorem 2, and the last equality follows from Proposition 7. Comparing Eqs. (58a)–(58c), we obtain a strict communication power hierarchy among different classes of decoders in the dense coding task, whenever Ψ_A is *mixed* and thus its quantum entropy is strictly positive:

$$C_{\mathfrak{E}_g,\mathfrak{D}_{\nleftrightarrow}}(\Psi_{AF}) < C_{\mathfrak{E}_g,\mathfrak{D}_\rightarrow}(\Psi_{AF}) < C_{\mathfrak{E}_g,\mathfrak{D}_{\rm glb}}(\Psi_{AF}). \tag{59}$$

B. Dense coding power of coherence

When G is a group of unitaries diagonal in a given basis $\{|b\rangle\}$ of system A (i.e., it is a subgroup of commuting unitaries), $\mathcal{G}$ becomes the completely dephasing channel $\Delta(\rho_A) = \sum_b \langle b|\,\rho_A\,|b\rangle\,|b\rangle\langle b|$ for all $\rho_A \in \mathscr{D}(\mathcal{H}_A)$ and we recover the intensively studied resource theory of coherence [112]. In this case, the encoders do not change the diagonal elements of the quantum state on the given basis

$\mathcal{B}$ but only affect the off-diagonal elements. In investigating the dense coding task under this resource theory we ask how much classical information can be encoded into quantum coherence resources. We have

$$C_{\mathfrak{E}_g,\mathfrak{D}_{\leftrightarrow}}(\Psi_{AF}) = H(\Delta(\Psi_A)) - H(\Psi_A), \tag{60a}$$

$$C_{\mathfrak{E}_g,\mathfrak{D}_{\rightarrow}}(\Psi_{AF}) = H(\Delta(\Psi_A)), \tag{60b}$$

$$C_{\mathfrak{E}_g,\mathfrak{D}_{\rm glb}}(\Psi_{AF}) = H(\Delta(\Psi_A)), \tag{60c}$$

where the first equality follows from a special case of Theorem 3, the second equality follows from Theorem 2, and the last equality follows from Proposition 7 and the fact that [115, Theorem 4]

$$D\left(\Psi_{AF} \| \Delta_A(\Psi_{AF})\right) = H(\Delta(\Psi_A)), \tag{61}$$

whenever Ψ_{AF} is pure. The fact that $C_{\mathfrak{E}_g,\mathfrak{D}_{\rightarrow}}(\Psi_{AF}) = C_{\mathfrak{E}_g,\mathfrak{D}_{\rm glb}}(\Psi_{AF})$ remarkably shows that global decoding has no advantage over one-way LOCC decoding for the dense coding task within the resource theory of quantum coherence. When the reduced density Ψ_A is diagonal, $C_{\mathfrak{E}_g,\mathfrak{D}_{\leftrightarrow}}(\Psi_{AF})$ evaluates to 0, indicating that incoherent quantum states have no communication power under our setting.

Indeed, the above discussion can be applied even when group G is the one-dimensional group $\mathbb{R}$ in the following case. Consider a diagonal Hermitian operator H whose diagonal elements are different. Then, we consider the unitary representation of $\mathbb{R}$ as $t \mapsto e^{itH}$. Each one-dimensional space generated by a diagonal element is a different irreducible component. Hence, we can apply the above discussion. When each diagonal element of H is an integer, the above can be considered as the unitary representation of the compact group $[0, 2\pi)$.

C. Two-mode squeezed vacuum state

We apply the discussion for dense coding power of coherence to the case when the preshared quantum state Ψ_{AF} is a two-mode squeezed vacuum (TMSV) state, which is given as

$$|\Psi\rangle_{AF} = \sum_{n=0}^{\infty} \sqrt{N^n/(N+1)^{n+1}}\, |n\rangle_A\, |n\rangle_F\,, \tag{62}$$

where n is the mean photon number and N is the average energy constraint. We compare our result with Ref. [134, Sec. IV], which addressed the special case of dense coding capacity $C_{\mathfrak{E}_g,\mathfrak{D}_{\rm glb}}(\Psi_{AF})$ using global measurements in this example. In this case, similar to Ref. [61], the encoder is given by the application of time evolution $U(t)$, where $U(t)$ is defined as $\exp(it\sum_{n=0}^{\infty} n|n\rangle\langle n|)$. Therefore, this problem can be considered a special case of the dense coding power of coherence.

In contrast, from the practical viewpoint, we impose the locality condition to our decoder. Then, state Ψ_{AF} defined in Eq. (62) satisfies the relation $H(\Delta(\Psi_A)) = H(\Psi_A) = (N+1)\log(N+1) - N\log N$. Hence, Eqs. (60a)–(60c) are simplified as

$$C_{\mathfrak{E}_g,\mathfrak{D}_{\leftrightarrow}}(\Psi_{AF}) = 0, \tag{63a}$$

$$C_{\mathfrak{E}_g,\mathfrak{D}_{\rightarrow}}(\Psi_{AF}) = (N+1)\log(N+1) - N\log N, \tag{63b}$$

$$C_{\mathfrak{E}_g,\mathfrak{D}_{\rm glb}}(\Psi_{AF}) = (N+1)\log(N+1) - N\log N. \tag{63c}$$

That is, the comparison between Eqs. (63b) and (63b) shows that the dense coding capacity given in Ref. [134, Sec. IV] can be attained by a one-way LOCC decoder, which does not need any joint measurement across the receiving system and the system of the helper. Furthermore, due to Eq. (63a), if only local decoders $\mathfrak{D}_{\leftrightarrow}$ are available, Alice cannot transmit classical information to Bob via the quantum state Ψ_{AF} in Eq. (62). This fact shows the importance of one-way LOCC, i.e., one-way LOCC extensively improves the communication speed. Since the TMSV state is a physically implementable system and the encoding can be implemented by a simple Hamiltonian $\sum_{n=0}^{\infty} n|n\rangle\langle n|$, this is a useful example to clarify the merit of the one-way LOCC decoder.

D. Dense coding power of Schur duality

Assume now that $\mathcal{H}_A = \mathcal{H}^{\otimes N}$, where $\mathcal{H}$ is a d-dimensional Hilbert space and N is the number of identical parties. That is, A is an N-partite system with equal local dimensions d. Group $\mathscr{U}(\mathcal{H})$ has the unitary representation $\{U^{\otimes N}\}_{U\in\mathscr{U}(\mathcal{H})}$ on $\mathcal{H}_A$. Let $S(N)$ be the set of permutations $\pi : [N] \to [N]$. Let $\pi \in S(N)$ be a permutation, and let W_π be the permutation unitary in $\mathcal{H}_A$ induced by π. Such a unitary reorders the output systems according to π. In this case, $\mathcal{H}_A$ is decomposed to

$$\mathcal{H} = \bigoplus_k \mathcal{U}_k \otimes \mathcal{V}_k, \tag{64}$$

where $\mathcal{U}_k$ is an irreducible space of group $\mathscr{U}(\mathcal{H})$ and $\mathcal{V}_k$ is an irreducible space of the permutation group $S(N)$.

When group G is chosen as $\mathscr{U}(\mathcal{H})$ in a similar way to Ref. [91, Sec. IV-D], the multiplicity-free condition is not satisfied because the dimension of $\mathcal{V}_k$ shows the multiplicity of representation $\mathcal{U}_k$. When group G is chosen as $S(N)$, the multiplicity-free condition is not satisfied because the dimension of $\mathcal{U}_k$ shows the multiplicity of representation $\mathcal{V}_k$. However, when group G is chosen as $\mathscr{U}(\mathcal{H}) \times S(n)$, the multiplicity-free condition is satisfied because the spaces $\mathcal{U}_k \otimes \mathcal{V}_k$ are different irreducible spaces. We emphasize that Ref. [91] considered mainly the quantum channel coding satisfying group symmetry but did not investigate the dense coding problem under locality conditions.

In the following, we choose $d = 2$ and $\Psi_A = \rho^{\otimes N}$ with ρ being a density matrix on $\mathcal{H} = \mathbb{C}^2$ as an example to evaluate the dense coding capacities $C_{\mathfrak{E}_g,\mathfrak{D}_{\leftrightarrow}}(\Psi_{AF})$, $C_{\mathfrak{E}_g,\mathfrak{D}_{\rightarrow}}(\Psi_{AF})$, and $C_{\mathfrak{E}_g,\mathfrak{D}_{\mathrm{glb}}}(\Psi_{AF})$. Also, we assume that the eigenvalues of ρ are p and $1-p$ with $1 \geq 2p$. Then, $H(\Psi_A) = H(F)_\xi = Nh(p)$, where h is the binary entropy. In this case, the irreducible space is labeled by $k = 0, \ldots, \lfloor N/2 \rfloor$. The dimension of the irreducible space $\mathcal{H}_k$ is $(N+1-k)(\binom{N}{k} - \binom{N}{k-1})$, where $\binom{N}{-1}$ is defined to be 0. Then we have

$$P_K(k) = \left(\binom{N}{k} - \binom{N}{k-1}\right) q_k, \tag{65}$$

where $q_k := [p^k(1-p)^{N-k+1} - p^{N-k+1}(1-p)^k]/(1-2p)$. Hence,

$$H(K)_\xi = -\sum_{k=0}^{\lfloor N/2 \rfloor} P_K(k) \log P_K(k), \tag{66}$$

$$H(A)_\xi = -\sum_{k=0}^{\lfloor N/2 \rfloor} P_K(k) \log \frac{q_k}{N+1-k}. \tag{67}$$

Therefore,

$$C_{\mathfrak{E}_g,\mathfrak{D}_{\leftrightarrow}}(\Psi_{AF}) = -\sum_{k=0}^{\lfloor N/2 \rfloor} P_K(k) \log \frac{q_k}{N+1-k} - Nh(p), \tag{68a}$$

$$C_{\mathfrak{E}_g,\mathfrak{D}_{\rightarrow}}(\Psi_{AF}) = -\sum_{k=0}^{\lfloor N/2 \rfloor} P_K(k) \log \frac{q_k}{N+1-k}, \tag{68b}$$

$$\begin{aligned} C_{\mathfrak{E}_g,\mathfrak{D}_{\mathrm{glb}}}(\Psi_{AF}) &= H(A)_\xi + H(F|K)_\xi \\ &= \sum_{k=0}^{\lfloor N/2 \rfloor} P_K(k) \log(N+1-k) \left(\binom{N}{k} - \binom{N}{k-1}\right) + Nh(p). \end{aligned} \tag{68c}$$

We visualize these dense coding capacities as functions of N in Fig. 6. From this figure, we can see the dense coding power hierarchy of different decoders: the less the locality constraint on the decoders, the larger the corresponding dense coding capacity.

VI. EXTENSION TO A NONQUANTUM PRESHARED STATE

In the above dense coding framework, we assume that the preshared resource on the bipartite system $\mathcal{H}_{AF}$ is a bipartite quantum state (positive semidefinite operator with unit trace). However, if the decoders are limited to separable measurements $\mathfrak{D}_{\mathrm{sep}}$ or PPT measurements $\mathfrak{D}_{\mathrm{ppt}}$, it is theoretically possible that the state on the bipartite system $\mathcal{H}_{AF}$ is not a quantum state. That is, we can loosen the positive semidefiniteness constraint.

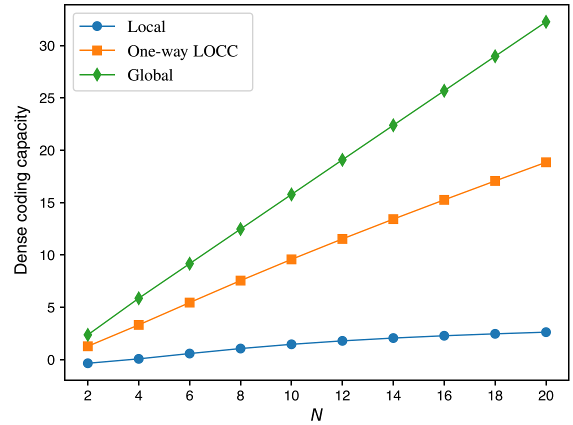

FIG. 6. Three dense coding capacities—$C_{\mathfrak{E}_g,\mathfrak{D}_{\leftrightarrow}}(\Psi_{AF})$ with local encoders, $C_{\mathfrak{E}_g,\mathfrak{D}_{\rightarrow}}(\Psi_{AF})$ with one-way LOCC decoders, and $C_{\mathfrak{E}_g,\mathfrak{D}_{\mathrm{glb}}}(\Psi_{AF})$ with global decoders—as functions of N, where N is the number of identical parties. We set parameter $p = 1/4$.

As a demonstrative example, we assume that the available decoders are separable measurements $\mathfrak{D}_{\mathrm{sep}}$. We denote the cone composed of separable operators on the bipartite system $\mathcal{H}_{AF}$ by SEP and the dual cone by SEP*. Then, we define the set $\mathcal{S}(\mathrm{SEP}^*) := \{\rho \in \mathrm{SEP}^* \mid \mathrm{Tr}\,\rho = 1\}$. For the preshared "resource" $\rho_{AF} \in \mathcal{S}(\mathrm{SEP}^*)$, we can analogously define the dense coding capacities

$$C^{\epsilon}_{\mathfrak{E},\mathfrak{D}}(\rho_{AF}), \quad C_{\mathfrak{E},\mathfrak{D}}(\rho_{AF}), \quad C^{\dagger}_{\mathfrak{E},\mathfrak{D}}(\rho_{AF}), \tag{69}$$

$$\widetilde{C}^{\epsilon}_{\mathfrak{E},\mathfrak{D}_{\rightarrow}}(\rho_{AF}), \quad \widetilde{C}_{\mathfrak{E},\mathfrak{D}_{\rightarrow}}(\rho_{AF}), \quad \widetilde{C}^{\dagger}_{\mathfrak{E},\mathfrak{D}_{\rightarrow}}(\rho_{AF}), \tag{70}$$

where $\mathfrak{E} \in \{\mathfrak{E}_g, \mathfrak{E}_{\mathrm{cp}}, \mathfrak{E}_{\mathrm{ppt}}, \mathfrak{E}_p\}$ and $\mathfrak{D} \in \{\mathfrak{D}_{\rightarrow}, \mathfrak{D}_{\leftrightarrow}, \mathfrak{D}_{\mathrm{sep}}\}$ in the same way as Sec. IV.

Similarly, we denote the cone composed of PPT operators on the bipartite system $\mathcal{H}_{AF}$ by PPT and the dual cone by PPT*. Then, we define the set $\mathcal{S}(\mathrm{PPT}^*) := \{\rho \in \mathrm{PPT}^* \mid \mathrm{Tr}\,\rho = 1\} \subset \mathcal{S}(\mathrm{SEP}^*)$. For the preshared "resource" $\rho'_{AF} \in \mathcal{S}(\mathrm{PPT}^*)$, we can define in the same way the dense coding capacities

$$C^{\epsilon}_{\mathfrak{E},\mathfrak{D}_{\mathrm{ppt}}}(\rho'_{AF}), \quad C_{\mathfrak{E},\mathfrak{D}_{\mathrm{ppt}}}(\rho'_{AF}), \quad C^{\dagger}_{\mathfrak{E},\mathfrak{D}_{\mathrm{ppt}}}(\rho'_{AF}), \tag{71}$$

where $\mathfrak{E} \in \{\mathfrak{E}_g, \mathfrak{E}_{\mathrm{cp}}, \mathfrak{E}_{\mathrm{ppt}}\}$.

Regarding the above nonquantum preshared state extension, we have the following strong converse theorem, much like the strong converse parts of Theorems 1 and 2. See Appendix F for the proof.

Theorem 4 (Strong converse part). *For $\rho_{AF} \in \mathcal{S}(\mathrm{SEP}^*)$ and $\rho'_{AF} \in \mathcal{S}(\mathrm{PPT}^*)$, it holds that*

$$\widetilde{C}^{\dagger}_{\mathfrak{E}_p,\mathfrak{D}_{\rightarrow}}(\rho_{AF}) \leq H(\mathcal{G}(\rho_A)), \tag{72}$$

$$C^{\dagger}_{\mathfrak{E}_p,\mathfrak{D}_{sep}}(\rho_{AF}) \le H(\mathcal{G}(\rho_A)), \tag{73}$$

$$C^{\dagger}_{\mathfrak{E}_{ppt},\mathfrak{D}_{ppt}}(\rho'_{AF}) \le H(\mathcal{G}(\rho'_A)). \tag{74}$$

However, we are not able to prove the direct part as that of Theorem 2 for the nonquantum preshared state extension except that some additional conditions are satisfied. The result is summarized in the following theorem. See Appendix F for the proof.

Theorem 5 (Direct part)**.** *Assume that the (projective) unitary representation U on $\mathcal{H}_A$ satisfies Assumption 1 (the multiplicity-free condition). Given $\rho_{AF} \in \mathcal{S}(\mathrm{SEP}^*)$ and $\rho'_{AF} \in \mathcal{S}(\mathrm{PPT}^*)$, we choose purifications Ψ_{AF} and Ψ_{AF} of ρ_A and ρ'_A, respectively. It holds that*

(a) if there exists a trace-preserving positive operation $\mathcal{E}_F \in \mathscr{C}(F \to F)_p$ such that $\mathcal{E}_F(\rho_{AF}) = \Psi_{AF}$, we have

$$H(\mathcal{G}(\rho_A)) \le C_{\mathfrak{E}_g,\mathfrak{D}_\to}(\rho_{AF}); \tag{75}$$

(b) if there exists a trace-preserving operation $\mathcal{E}'_F \in \mathscr{C}(F \to F)_{ppt}$ such that $\mathcal{E}'_F(\rho'_{AF}) = \Psi'_{AF}$, we have

$$H(\mathcal{G}(\rho'_A)) \le C_{\mathfrak{E}_g,\mathfrak{D}_\to}(\rho'_{AF}). \tag{76}$$

VII. CONCLUSION

In this paper, we have investigated practical dense coding thoroughly by imposing locality restrictions on decoders and symmetry restrictions on encoders within the resource theory of asymmetry framework. In this task, the preshared entangled state is fixed, the encoding operations are constrained by the resource theory of asymmetry, and the decoding measurements are restricted to local measurements. When the group representation characterizing the resource theory of asymmetry satisfies the multiplicity-free condition, we have derived both one-shot and asymptotic optimal achievable transmission rates of the dense coding task. What is more, we have studied the ultimate limit on the transmission rate when the encoding operations are relaxed to the most general operations allowed in the framework of GPT as superquantum encoders, and a particular locality condition is imposed on the decoding measurements. Our results revealed that this relaxation does not improve the transmission rate. Furthermore, we have shown that the same conclusion holds even when the initial state is not a quantum state but satisfies a particular fundamental condition.

Many interesting problems remain open. Firstly, we have imposed the multiplicity-free condition in Assumption 1 when proving the direct part of Theorem 2. It would be interesting to relax this condition. Secondly, we may consider the case when a specific locality condition, for example, the separability condition, is imposed on the initial state, and a class of superquantum measurements is allowed as the decoding measurement. Under this condition, the encoder can be relaxed to the class $\mathfrak{E}_{\mathrm{ppt}}$ or $\mathfrak{E}_p$. It is challenging to clarify whether this relaxation can yield higher transmission rates or not. Thirdly, under the same constraints for encoders and decoders, we can consider the dense coding capacities where any preshared entangled state is allowed between Alice and Fred, but the communication channel between Alice and Bob is a noisy quantum channel. It is interesting to clarify whether superquantum encoders can enhance the capacity in this setting. Lastly, we have imposed encoder constraints by using condition (1), which is related to the group symmetry. It is possible to consider the same setting with encoders given by CPTP maps $\mathscr{C}(A \to A)_{\mathrm{cp}}$, trace-preserving and positive maps $\mathscr{C}(A \to A)_p$, or $\mathscr{C}(A \to A)_{\mathrm{ppt}}$ without condition (1). It is another interesting problem to derive the capacity in this setting because our method does not work in this extended setting.

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APPENDIX A: USEFUL LEMMAS AND TOOLS

The following lemmas are intensively applied in our proofs.

Lemma 9 (Audenaert *et al.* [135])**.** *Let A and B be two non-negative Hermitian matrices. Denote by $\{A \ge B\}$ and $\{A > B\}$ the projectors onto the spaces spanned by the eigenvectors of $A - B$ with non-negative and positive eigenvalues, respectively. For arbitrary $s \in [0, 1]$, it holds that*

$$\mathrm{Tr}[A\{B \ge A\}] + \mathrm{Tr}[B\{A > B\}] \le \mathrm{Tr}[A^{1-s}B^s]. \tag{A1}$$

Lemma 10 (Pinching lemma; Lemma 3.10 of Ref. [103])**.** *Let $\{\Lambda_m\}_m$ be a POVM of size d in the Hilbert space $\mathcal{H}$. For arbitrary quantum state $\rho \in \mathscr{D}(\mathcal{H})$, it holds that*

$$d\left(\sum_m \Lambda_m \rho \Lambda_m\right) \ge \rho. \tag{A2}$$

1. Operator convex and concave functions

We briefly recover the definitions of operator convex and concave functions here to make the paper self-contained. We refer interested readers to Ref. [102, Section 2.5] and Ref. [103, Appendix A.4] for more details. A function $f : \mathbb{R}^+ \to \mathbb{R}$ is called *operator convex* if, for arbitrary $M, N \in \mathscr{P}(\mathcal{H})$ and $\lambda \in [0,1]$, it holds that

$$f(\lambda M + (1-\lambda)N) \le \lambda f(M) + (1-\lambda) f(N). \tag{A3}$$

Conversely, if Eq. (A3) holds with the inequality reversed then function f is called *operator concave*. In the following lemma, we give a concrete example of operator concave functions frequently used in this paper.

Lemma 11. *The function $t \mapsto t^\alpha$ is operator convex when $\alpha \in [1,2]$ and operator concave when $\alpha \in (0,1]$.*

Finally, we present the classical-quantum channel coding theorem. We consider a set of classical inputs $\mathcal{X}$. For an element x, we have a state W_x on a quantum system $\mathcal{H}$.

Lemma 12 (Ref. [136, Eq. (9)]). *We consider n uses of a CQ channel $x \mapsto W_x$. Given an integer M and a distribution P_X on $\mathcal{X}$, we define the average state $\overline{W} := \int_{\mathcal{X}} W_x P_X(dx)$. Then, there exist M elements $x_1, \ldots, x_M$ of $\mathcal{X}^n$ and a decoder POVM $\{\Pi_j\}_{j=1}^M$ such that*

$$\frac{1}{M}\sum_{j=1}^{M} \operatorname{Tr} W_{x_i}^{(n)}(I - \Pi_j) \le 4M^s \int_{\mathcal{X}} \operatorname{Tr} W_x^{1-s}\overline{W}^s P_X(dx) \tag{A4}$$

with an arbitrary $s \in [0,1]$.

APPENDIX B: PROOF OF PROPOSITION 2

The first inequality follows by definition. To show the second inequality, note that $H(\mathcal{G}(\rho))$ upper bounds $A_G(\rho)$ due to the concavity of quantum entropy:

$$\begin{aligned} A_G(\rho) &= \max_{\rho=\sum_x p_X(x)|\psi_x\rangle\langle\psi_x|} \sum_x p_X(x) H(\mathcal{G}(\psi_x)) \\ &\le \max_{\rho=\sum_x p_X(x)|\psi_x\rangle\langle\psi_x|} H\left(\mathcal{G}\left(\sum_x p_X(x)\psi_x\right)\right) \\ &= H(\mathcal{G}(\rho)). \end{aligned} \tag{B1}$$

This yields

$$\begin{aligned} A_G^\infty(\rho) &:= \limsup_{n\to\infty} \frac{1}{n} A_G(\rho^{\otimes n}) \le \lim_{n\to\infty}\frac{1}{n} H(\mathcal{G}^{\otimes n}(\rho^{\otimes n})) \\ &= H(\mathcal{G}(\rho)), \end{aligned} \tag{B2}$$

where the last equality follows from the fact that the quantum entropy is additive with respect to the tensor product.

APPENDIX C: PROOF OF THEOREM 1

Proof. Equation (45a) follows by definition.

The first inequality in Eq. (45b) follows by definition. Now we show the second inequality in Eq. (45b), which is commonly called the weak converse bound. Consider an enhanced dense coding protocol depicted in Fig. 5. By definition, we must search exhaustively over all possible POVMs in the environment to optimize the quantity $\widetilde{C}^{\mathfrak{E}}_{\mathfrak{E},\mathfrak{D}_\to}(\Psi_{AF})$, which is notoriously difficult. Luckily, it can be shown that Fred can restrict measurements to rank-one POVMs yet still achieve the same information transmission performance [119]. On the other hand, rank-one POVMs at Fred's side are in one-to-one correspondence with pure state decompositions of Ψ_A by the Schödinger-Hughston-Jozsa-Wootters theorem [137,138]

$$\Psi_A = \sum_x p_X(x) |\psi_A^x\rangle\langle\psi_A^x|, \tag{C1}$$

where $p_X(x)$ is a probability distribution and $\{|\psi_A^x\rangle\}_x$ is a set of pure states (not necessarily orthonormal). As a result, the task becomes how well Alice and Bob can encode classical information using the pure state ensemble $\{p_X(x), \psi_A^x\}$ on average. For each conditional state ψ_A^x, Alice performs a conditional encoding operation $\mathcal{E}^{m|x}_{A\to A} \in \mathfrak{E}$. In the single-shot case, the conditional mutual information between Alice's message and Bob' state is evaluated as

$$\begin{aligned} &\sum_x P_X(x)\frac{1}{|\mathcal{M}|}\sum_{m\in\mathcal{M}} D\left(\mathcal{E}^m_{A\to A}(\psi_A^x)\middle\|\frac{1}{|\mathcal{M}|}\sum_{m'\in\mathcal{M}}\mathcal{E}^{m'}_{A\to A}(\psi_A^x)\right) \\ &\le \sum_x P_X(x)\frac{1}{|\mathcal{M}|}\sum_{m\in\mathcal{M}} D\left(\mathcal{E}^m_{A\to A}(\psi_A^x)\middle\|\mathcal{G}(\psi_A^x)\right) \\ &\overset{(a)}{=} \sum_x P_X(x)\frac{1}{|\mathcal{M}|}\sum_{m\in\mathcal{M}} D\left(\mathcal{E}^m_{A\to A}(\psi_A^x)\middle\|\mathcal{E}^m_{A\to A}\circ\mathcal{G}(\psi_A^x)\right) \\ &\overset{(b)}{\le} \sum_x P_X(x)\frac{1}{|\mathcal{M}|}\sum_{m\in\mathcal{M}} D\left(\psi_A^x\middle\|\mathcal{G}(\psi_A^x)\right) \\ &= \sum_x P_X(x) D\left(\psi_A^x\middle\|\mathcal{G}(\psi_A^x)\right) \\ &\le A_G(\Psi_A) \\ &\le A_G^\infty(\Psi_A), \end{aligned} \tag{C2}$$

where (a) follows from the fact that $\mathcal{E}^{m|x}_{A\to A} \in \mathfrak{E}$ and (b) follows from the data processing inequality for the relative entropy with respect to the trace-preserving positive operations [139, Theorem 1]. Hence, when state $\Psi_A^{\otimes n}$ is given, the conditional mutual information between Alice's message and Bob's state is upper bounded by $nA_G^\infty(\Psi_A)$. Combining Fano's inequality [140], we can show the inequality $\widetilde{C}_{\mathfrak{E},\mathfrak{D}_\to}(\Psi_{AF}) \le A_G^\infty(\Psi_A)$.

The first inequality in Eq. (45c) follows by definition. Now we show the second inequality in Eq. (45c), which is commonly called the strong converse bound. It can be shown by use of a metaconverse technique originally invented in Ref. [141] and further investigated in Ref. [142, Chapter 3] (see also Refs. [143–145] for more applications of this technique). Roughly speaking, this metaconverse method guarantees that a quantum divergence satisfying certain reasonable properties induces an upper bound on the success probability of the communication protocol. Here we adopt the Petz Rényi divergence $\overline{D}_\alpha$ [101], which meets all required properties (cf. Sec. III B). In Proposition 13 below (which will be proved shortly), we upper bound the success probability of any one-shot enhanced code $\mathcal{C} \in (\mathfrak{E}, \mathfrak{D}_\rightarrow)$ in terms of the Rényi entropy, then the strong converse bound follows by considering block coding. More precisely, since $\lim_{\alpha\to 1} \overline{H}_{2-\alpha}(\mathcal{G}(\Psi_A)) = H(\mathcal{G}(\Psi_A))$ and $\overline{H}_\alpha$ is continuous and monotonically decreasing in α, Eq. (C3) guarantees that, for arbitrary $\log|\mathcal{C}| > H(\mathcal{G}(\Psi_A))$, there exists some $\alpha \in (1,2)$ for which the exponent $[(1-\alpha)/\alpha](\log|\mathcal{C}| - \overline{H}_{2-\alpha}(\mathcal{G}(\Psi_A)))$ is strictly positive, resulting in the success probability decaying exponentially fast to 0. This concludes that $H(\mathcal{G}(\Psi_A))$ is a strong converse bound. ■

Proposition 13. *Let $\mathfrak{E} \in \{\mathfrak{E}_g, \mathfrak{E}_{cp}, \mathfrak{E}_{ppt}, \mathfrak{E}_p\}$. Any enhanced dense coding code $\mathcal{C} \in (\mathfrak{E}, \mathfrak{D}_\rightarrow)$ as illustrated in Fig. 5 obeys the following bound for arbitrary $\alpha \in (1,2)$:*

$$s(\mathcal{C}) \leq \exp\left\{\frac{\alpha-1}{\alpha}\left(\overline{H}_{2-\alpha}(\mathcal{G}(\Psi_A)) - \log|\mathcal{C}|\right)\right\} \tag{C3}$$

with $\overline{H}_\alpha$ the Rényi entropy defined in Eq. (10).

Proof. Step 1. We first introduce the notation used in the proof of Eq. (C3). Let $\{\Lambda^x\}_{x\in\mathcal{X}}$ be the measurement carried out by Fred. Set $p_X(x) := \mathrm{Tr}[\Lambda^x\Psi_F]$ and $\psi_A^x := \mathrm{Tr}_F[(\mathbb{1}_A \otimes \Lambda^x)\Psi_{AF}]/p_X(x)$. We define the two quantum states

$$\rho_{MXA} := \frac{1}{|\mathcal{M}|}\sum_m |m\rangle\langle m|_M \otimes \sum_x p_X(x)|x\rangle\langle x|_X \otimes \mathcal{E}^{m|x}(\psi_A^x), \tag{C4}$$

$$\sigma_{MXA} := \pi_M \otimes \sum_x p_X(x)|x\rangle\langle x|_X \otimes \mathcal{G}(\psi_A^x), \tag{C5}$$

where ρ_{MXA} serves as a *test state*. For given ψ_A^x, we choose a pure state decomposition as $\psi_A^x = \sum_y P_{Y|X}(y|x)\psi_A^{x,y}$. Then, we have

$$\rho_{MXYA} := \frac{1}{|\mathcal{M}|}\sum_m |m\rangle\langle m|_M \otimes \sum_{x,y} p_{XY}(x,y)|x,y\rangle\langle x,y|_X \otimes \mathcal{E}^{m|x}(\psi_A^{x,y}), \tag{C6}$$

$$\sigma_{MXYA} := \pi_M \otimes \sum_{x,y} p_{XY}(x,y)|x,y\rangle\langle x,y|_{XY} \otimes \mathcal{G}(\psi_A^{x,y}). \tag{C7}$$

Given a code $(\{\mathcal{E}^{m|x}\}, \{\Gamma^{\widehat{m}|x}\})$ depending on x, the positive operator

$$T := \sum_m \sum_x |m\rangle\langle m|_M \otimes |x\rangle\langle x|_X \otimes \Gamma^{m|x} \tag{C8}$$

satisfies

$$\mathrm{Tr}\, T\rho_{MXA} = \frac{1}{|\mathcal{M}|}\sum_m p_{\widehat{M}M}(m|m) = s(\mathcal{C}), \tag{C9}$$

$$\mathrm{Tr}\, T\sigma_{MXA} = \frac{1}{|\mathcal{M}|}\sum_x p_X(x)\,\mathrm{Tr}\left[\sum_m \Gamma^{m|x}\mathcal{G}(\psi_A^x)\right] \stackrel{(a)}{=} \frac{1}{|\mathcal{M}|}\sum_x p_X(x)\,\mathrm{Tr}[\mathcal{G}(\psi_A^x)] = \frac{1}{|\mathcal{M}|}, \tag{C10}$$

where (a) follows from the fact that, for each x, the conditional decoding operation $\{\Gamma^{m|x}\}_{m\in\mathcal{M}}$ forms a POVM. Applying the data processing inequality of $\overline{D}_\alpha$ to the binary measurement $\{T, I-T\}$, we have

$$s(\mathcal{C})^\alpha\left(\frac{1}{|\mathcal{C}|}\right)^{1-\alpha} + (1-s(\mathcal{C}))^\alpha\left(1-\frac{1}{|\mathcal{C}|}\right)^{1-\alpha} \leq e^{(\alpha-1)\overline{D}_\alpha(\rho_{MXA}\|\sigma_{MXA})}. \tag{C11}$$

Step 2. Now we show the key result (C3). Consider the following chain of inequalities:

$$s(\mathcal{C})^\alpha|\mathcal{C}|^{\alpha-1} = s(\mathcal{C})^\alpha\left(\frac{1}{|\mathcal{C}|}\right)^{1-\alpha} \stackrel{(a)}{\leq} s(\mathcal{C})^\alpha\left(\frac{1}{|\mathcal{C}|}\right)^{1-\alpha} + (1-s(\mathcal{C}))^\alpha\left(1-\frac{1}{|\mathcal{C}|}\right)^{1-\alpha} \tag{C12}$$

$$
\begin{aligned}
&\overset{\text{(b)}}{\leq} e^{(\alpha-1)\overline{D}_\alpha(\rho_{MXA}\|\sigma_{MXA})} \\
&\overset{\text{(c)}}{\leq} e^{(\alpha-1)\overline{D}_\alpha(\rho_{MXYA}\|\sigma_{MXYA})} \\
&\overset{\text{(d)}}{\leq} \frac{1}{|\mathcal{M}|}\sum_m\sum_{x,y} p_{XY}(x,y) e^{(\alpha-1)\overline{D}_\alpha\left(\mathcal{E}^{m|x}(\psi_A^x)\middle\|\mathcal{G}(\psi_A^x)\right)} \\
&\overset{\text{(e)}}{=} \frac{1}{|\mathcal{M}|}\sum_m\sum_{x,y} p_{XY}(x,y)\int_G e^{(\alpha-1)\overline{D}_\alpha\left(U_g\mathcal{E}^{m|x}(\psi_A^{x,y})U_g^\dagger\middle\|U_g\mathcal{G}(\psi_A^{x,y})U_g^\dagger\right)}\nu(dg) \\
&\overset{\text{(f)}}{=} \frac{1}{|\mathcal{M}|}\sum_m\sum_{xy} p_{XY}(x,y)\int_G e^{(\alpha-1)\overline{D}_\alpha\left(U_g\mathcal{E}^{m|x}(\psi_A^{x,y})U_g^\dagger\middle\|\mathcal{G}(\psi_A^{x,y})\right)}\nu(dg) \\
&= \frac{1}{|\mathcal{M}|}\sum_m\sum_{x,y} p_{XY}(x,y)\operatorname{Tr}[(U_g(\mathcal{E}^{m|x}(\psi_A^{x,y}))U_g^\dagger)^\alpha\mathcal{G}(\psi_A^{x,y})^{1-\alpha}]\nu(dg) \\
&\overset{\text{(g)}}{\leq} \frac{1}{|\mathcal{M}|}\sum_m\sum_{x,y} p_{XY}(x,y)\operatorname{Tr}[(U_g(\mathcal{E}^{m|x}(\psi_A^{x,y}))U_g^\dagger)\mathcal{G}(\psi_A^{x,y})^{1-\alpha}]\nu(dg) \\
&= \frac{1}{|\mathcal{M}|}\sum_m\operatorname{Tr}\sum_{x,y} p_{XY}(x,y)\left[\left(\int_G U_g(\mathcal{E}^{m|x}(\psi_A^{x,y}))U_g^\dagger\right)\nu(dg)\mathcal{G}(\psi_A^{x,y})^{1-\alpha}\right] \\
&= \frac{1}{|\mathcal{M}|}\sum_m\operatorname{Tr}\sum_{x,y} p_{XY}(x,y)[\mathcal{G}\circ\mathcal{E}^{m|x}(\psi_A^{x,y})\mathcal{G}(\psi_A^{x,y})^{1-\alpha}] \\
&\overset{\text{(h)}}{=} \operatorname{Tr}\left[\sum_{x,y} p_{XY}(x,y)\mathcal{G}(\psi_A^{x,y})^{2-\alpha}\right] \\
&\overset{(i)}{\leq} \operatorname{Tr}\left[\mathcal{G}\left(\sum_{x,y} p_{XY}(x,y)\psi_A^{x,y}\right)^{2-\alpha}\right] \\
&= e^{(\alpha-1)\overline{H}_{2-\alpha}(\mathcal{G}(\Psi_A))}.
\end{aligned}
\tag{C13}
$$

Here

1. inequality (a) follows since the added term is nonnegative,
2. inequality (b) follows from Eq. (C11),
3. inequality (c) follows from the data processing inequality of $\overline{D}_\alpha$ for the partial trace operation Tr_Y,
4. inequality (d) follows from the joint convexity for $e^{(\alpha-1)\overline{D}_\alpha}$, which was proved in Ref. [102, Proposition 4.8],
5. equality (e) follows from the fact that $\overline{D}_\alpha$ is invariant with respect to the unitary channel,
6. equality (f) follows from the definition of $\mathcal{G}$,
7. inequality (g) follows from the inequality $x^\alpha \leq x$ for $x \in [0,1]$ and $\alpha > 1$,
8. equality (h) follows from the equation $\mathcal{G}\circ\mathcal{E}^{m|x} = \mathcal{G}$ since $\mathcal{E}^{m|x} \in \mathfrak{E}$, and
9. inequality (i) follows from the fact that $t \mapsto t^{2-\alpha}$ is operator concave when $\alpha \in [1,2)$ as given in Lemma 11.

Rearranging the above inequality leads to Eq. (C3). ■

APPENDIX D: PROOF OF THEOREM 2

Based on Proposition 2, Eq. (43), and Theorem 1, to show Theorem 2, it suffices to show the inequalities

$$
\begin{aligned}
C^\dagger_{\mathfrak{E}_{\mathrm{ppt}},\mathfrak{D}_{\mathrm{ppt}}}(\Psi_{AF}), C^\dagger_{\mathfrak{E}_p,\mathfrak{D}_{\mathrm{sep}}}(\Psi_{AF}) &\leq H(\mathcal{G}(\Psi_A)) \\
&\leq C_{\mathfrak{E}_g,\mathfrak{D}_\rightarrow}(\Psi_{AF}),
\end{aligned}
\tag{D1}
$$

The second inequality of Eq. (D1) is known as the direct part (or the achievability part), meaning that there exist encoding operations from $\mathfrak{E}_g$ and one-way LOCC decoding measurements from $\mathfrak{D}_\rightarrow$ for which the rate $H(\mathcal{G}(\Psi_A))$ is achievable. We show this direct part in Appendix D 1. Note that the proof of the one-shot direct part remains the most difficult part in this work. The first inequality of Eq. (D1) is known as the strong converse bound, meaning that $H(\mathcal{G}(\Psi_A))$ is an upper bound on all possible achievable coding rates even if coding rate and decoding error trade-offs are allowed. We show this strong converse part in Appendix D 2.

1. Direct part under the multiplicity-free condition

In this section, we prove the second inequality of Eq. (D1) under Assumption 1 (the multiplicity-free condition). We first present a one-shot direct part and then apply it to the asymptotic regime.

a. One-shot direct part

We first introduce some new notation used only for the proof of this one-shot direct part. With out loss of generality, the purification of the shared resourceful quantum state Ψ_A can be chosen with the form

$$|\Psi\rangle_{AF} := \frac{1}{\sqrt{N}} \sum_{n\in\mathcal{N}} |\psi_{A,n}\rangle |n\rangle_F , \tag{D2}$$

where $\mathcal{N}$ is some alphabet, $N \equiv |\mathcal{N}| \geq \text{rank}(\Psi_A)$ is the size of the alphabet, $\{|n\rangle\}$ is an orthonormal basis of F, and $\{|\psi_{A,n}\rangle\}$ is a set of pure states (not necessarily orthonormal) of A. Also, $\text{rank}(\Psi_A)$ expresses the rank of state Ψ_A. In this purification, system F is N dimensional. We remark that such uniform purification is always possible as long as $N \geq \text{rank}(\Psi_A)$ [146, Exercise 5.1.3]. Under this purification, we have $\Psi_A = (1/N)\sum_{n\in\mathcal{N}} |\psi_{A,n}\rangle\langle\psi_{A,n}|$. For each pure conditional state $\psi_{A,n}$, define its twirled version as $\overline{\rho}_{A,n} := \mathcal{G}(\psi_{A,n})$. Correspondingly, the twirled state of Ψ_A is

$$\overline{\rho}_A := \mathcal{G}(\Psi_A) = \frac{1}{N}\sum_{n\in\mathcal{N}} \mathcal{G}(|\psi_{A,n}\rangle\langle\psi_{A,n}|) = \frac{1}{N}\sum_{n\in\mathcal{N}} \overline{\rho}_{A,n}. \tag{D3}$$

Then, state ξ_{AF} defined in Eq. (47) of the main text has the equivalent expression

$$\xi_{AF} = \frac{1}{N}\sum_{n\in\mathcal{N}} \overline{\rho}_{A,n} \otimes |n\rangle\langle n|_F. \tag{D4}$$

Note that $\xi_A = \overline{\rho}_A$. ξ_{AF} can be obtained from $|\Psi\rangle_{AF}$ by first dephasing F in the orthonormal basis and then twirling system A via $\mathcal{G}$. We evaluate here various Petz-Rényi entropies of ξ_{AF} that are useful for later analysis:

$$\begin{aligned}\overline{H}_\alpha(F|A)_\xi &:= -\overline{D}_\alpha\left(\xi_{AF}\|\mathbb{1}_F\otimes\xi_A\right)\\ &= \frac{1}{1-\alpha}\log\frac{1}{N^\alpha}\sum_{n\in\mathcal{N}}\text{Tr}[\overline{\rho}_{A,n}^{\alpha}\overline{\rho}_A^{1-\alpha}],\end{aligned} \tag{D5a}$$

$$\begin{aligned}\overline{H}_\alpha(A|F)_\xi &:= -\overline{D}_\alpha\left(\xi_{AF}\|\mathbb{1}_A\otimes\xi_F\right)\\ &= \frac{1}{1-\alpha}\log\text{Tr}\left[\frac{1}{N}\sum_{n\in\mathcal{N}}\overline{\rho}_{A,n}^{\alpha}\right],\end{aligned} \tag{D5b}$$

$$\overline{H}_\alpha(A)_\xi := \frac{1}{1-\alpha}\log\text{Tr}\,\overline{\rho}_A^{\alpha}. \tag{D5c}$$

We focus on two convex functions $-s\overline{H}_{1+s}(AF)_\xi + s\overline{H}_{1-s}(F|A)_\xi$ and $-s\overline{H}_{1+s}(A)_\xi$). The maximum of them, i.e., $\max(-s\overline{H}_{1+s}(AF)_\xi + s\overline{H}_{1-s}(F|A)_\xi, -s\overline{H}_{1+s}(A)_\xi)$ is also a convex function. We define the Legendre transformation of the convex function as

$$\begin{aligned}\mathcal{L}_\xi(R) := \max_{0\leq s\leq 1} sR + \min(&s\overline{H}_{1+s}(AF)_\xi\\ &- s\overline{H}_{1-s}(F|A)_\xi, s\overline{H}_{1+s}(A)_\xi).\end{aligned} \tag{D6}$$

Now we are ready to state the one-shot direct coding theorem, which characterizes the one-shot ε-dense coding capacity $C^{\varepsilon}_{\mathfrak{E}_g,\mathfrak{D}_\to}(\Psi_{AF})$, where the available encoders are the unitary representations $\mathfrak{E}_g$ and the available decoders are the one-way LOCC measurements $\mathfrak{D}_\to$.

Theorem 6 (One-shot direct part). *Let $|\Psi\rangle_{AF}$ be a bipartite pure quantum state and $\varepsilon\in[0,1)$. When the (projective) unitary representation U on $\mathcal{H}_A$ satisfies Assumption 1 (the multiplicity-free condition), it holds that*

$$-\mathcal{L}_\xi^{-1}(-\log\varepsilon) \leq C^{\varepsilon}_{\mathfrak{E}_g,\mathfrak{D}_\to}(\Psi_{AF}), \tag{D7}$$

where $\mathcal{L}_\xi$ is defined in Eq. (D6) and the one-shot ε dense coding capacity $C^{\varepsilon}_{\mathfrak{E}_g,\mathfrak{D}_\to}(\Psi_{AF})$ is defined in Eq. (32).

Proof. We prove Theorem 6 in five steps.

Step 1. The following argument relies on the two-universal hash function elaborated in Ref. [147, Section 5.4]. We introduce a measurement induced by the two-universal hash function. Let $\mathcal{T}\subset\mathcal{N}$ be a strict subset of $\mathcal{N}$ and set $T := |\mathcal{T}|$. Let $\hat{F}:\mathcal{N}\to\mathcal{T}$ be a linear surjective two-universal hashing function. Then, we define the random variable $\hat{T} := \hat{F}(\hat{N})$, where $\hat{N}$ is the uniform random variable on $\mathcal{N}$. The hashing function f splits system F into T nonoverlapping subspaces $\mathcal{S}^{f,t}$ with the corresponding subspace projectors

$$\Pi^{f,t} := \sum_{n\in f^{-1}(t)} |n\rangle\langle n|_F. \tag{D8}$$

Note that $\{n\in f^{-1}(t)\}$ is of the same size for each t and is given by $L := N/T$. As such, each subspace $\mathcal{S}^{f,t}$ is L dimensional. Fix the pair (f,t). Let $\mathbf{Q}^{f,t} := \{Q^{f,t}_{A,n}\}_{n\in f^{-1}(t)}$ be a POVM on A. Define the δ function

$$\delta^{f,t} := 1 - \frac{1}{L}\sum_{n\in f^{-1}(t)}\text{Tr}[Q^{f,t}_{A,n}\overline{\rho}_{A,n}]. \tag{D9}$$

Roughly, $\delta^{f,t}$ quantifies how well the measurement $\mathbf{Q}^{f,t}$ detects the twirled states lying inside the subspace projected by $\Pi^{f,t}$. Choosing different pairs (f,t), we are able to construct a list of POVMs $\mathbf{Q}^{f,t}$ such that $Q^{f,t}_{A,n}$ and $\overline{\rho}_{A,n}$ are commutative for each n and the expected value (with respect to both f and t) of the δ function is upper bounded as

$$\begin{aligned}
\mathbb{E}_{\hat{F},\hat{T}}\delta^{F,T} &= 1-\mathbb{E}_{\hat{F}}\frac{1}{N}\sum_{n\in\mathcal{N}}\mathrm{Tr}[Q^{\hat{F},\hat{F}(n)}_{A,n}\overline{\rho}_{A,n}]\\
&\overset{\text{(a)}}{\le}\frac{1}{N}\sum_{n\in\mathcal{N}}\mathrm{Tr}[\overline{\rho}_{A,n}\{\overline{\rho}_{A,n}\ge L\overline{\rho}_A\}]\\
&\quad+\mathrm{Tr}[L\overline{\rho}_A\{\overline{\rho}_{A,n}<L\overline{\rho}_A\}]\\
&\overset{\text{(b)}}{\le}\frac{L^{s'}}{N}\sum_{n\in\mathcal{N}}\mathrm{Tr}[\overline{\rho}^{1-s'}_{A,n}\overline{\rho}^{s'}_A]\\
&\overset{\text{(c)}}{=}\frac{L^{s'}}{N}N^{1-s'}e^{s'\overline{H}_{1-s'}(F|A)_\xi}\\
&\overset{\text{(d)}}{=}T^{-s'}e^{s'\overline{H}_{1-s'}(F|A)_\xi},
\end{aligned}\tag{D10}$$

where $s'\in[0,1]$, (a) follows from the Hayashi-Nagaoka inequality [148], (b) follows from Lemma 9, which is a well-known inequality in hypothesis testing, (c) follows from Eq. (D5a), and (d) follows from $L=N/T$. Based on the same construction, we can estimate the following expectation with respect to both $\hat{F}$ and $\hat{T}$:

$$\begin{aligned}
&\mathbb{E}_{\hat{F},\hat{T}}\mathrm{Tr}\left[\left(\frac{1}{L}\sum_{n\in\hat{F}^{-1}(T)}\overline{\rho}_{A,n}\right)^{1+s}\right]\\
&=\mathbb{E}_{\hat{F}}\mathrm{Tr}\left[\frac{1}{N}\sum_{n\in\mathcal{N}}\overline{\rho}_{A,n}\left(\frac{1}{L}\overline{\rho}_{A,n}+\frac{1}{L}\sum_{n'(\neq n)\in\mathcal{N}:\hat{F}(n')=F(n)}\overline{\rho}_{A,n'}\right)^{s}\right]\\
&\overset{\text{(a)}}{\le}\mathrm{Tr}\left[\frac{1}{N}\sum_{n\in\mathcal{N}}\overline{\rho}_{A,n}\left(\frac{1}{L}\overline{\rho}_{A,n}+\mathbb{E}_{\hat{F}}\frac{1}{L}\sum_{n'(\neq n)\in\mathcal{N}:\hat{F}(n')=F(n)}\overline{\rho}_{A,n'}\right)^{s}\right]\\
&=\mathrm{Tr}\left[\frac{1}{N}\sum_{n\in\mathcal{N}}\overline{\rho}_{A,n}\left(\frac{1}{L}\overline{\rho}_{A,n}+\frac{L-1}{L}\overline{\rho}_A\right)^{s}\right]\\
&\le\mathrm{Tr}\left[\frac{1}{N}\sum_{n\in\mathcal{N}}\overline{\rho}_{A,n}\left(\frac{1}{L}\overline{\rho}_{A,n}+\overline{\rho}_A\right)^{s}\right]\\
&\overset{\text{(b)}}{\le}\mathrm{Tr}\left[\frac{1}{N}\sum_{n\in\mathcal{N}}\overline{\rho}_{A,n}\left(\frac{1}{L^s}\overline{\rho}^s_{A,n}+\overline{\rho}^s_A\right)\right]\\
&=\frac{1}{L^s}\mathrm{Tr}\left[\frac{1}{N}\sum_{n\in\mathcal{N}}\overline{\rho}^{1+s}_{A,n}\right]+\mathrm{Tr}[\overline{\rho}^{1+s}_A]\\
&\overset{\text{(c)}}{=}\frac{1}{L^s}e^{-s\overline{H}_{1+s}(A|F)_\xi}+e^{-s\overline{H}_{1+s}(A)_\xi}.
\end{aligned}\tag{D11}$$

Here $s\in[0,1]$, (a) follows from the concavity of $x\mapsto x^s$ when $s\in(0,1]$ as given in Lemma 11 and the Jensen inequality, (b) follows from $(x+y)^s\le x^s+y^s$ for $x,y>0$, and (c) follows from Eqs. (D5b) and (D5c).

Step 2. We prepare a useful lemma that holds under Assumption 1 (the multiplicity-free condition) as follows. This lemma will be shown in step 5 below. Assume that the (projective) unitary representation U on $\mathcal{H}_A$ is multiplicity-free. Then, the Hilbert space $\mathcal{H}_A$ can be decomposed as $\bigoplus_{k\in\mathcal{K}}\mathcal{H}_k$. Let $\{Q_{A,l}\}_{l=1}^L$ be a POVM on $\mathcal{H}_A$ such that each $Q_{A,l}$ is a projection onto the invariant subspace $\bigoplus_{k\in\mathcal{S}_l}\mathcal{H}_k$, where $\{\mathcal{S}_l\}_{l=1}^L$ are disjoint subsets of $\mathcal{K}$. That is, each $Q_{A,l}$ projects into the subspace $\bigoplus_{k\in\mathcal{S}_l}\mathcal{H}_k$.

Define the corresponding conditional projection operator in AF as

$$Q_{AF} := \sum_{l=1}^{L} Q_{A,l} \otimes |l\rangle\langle l|_F, \tag{D12}$$

where $\{|l\rangle\}_{l=1}^{L}$ is an orthonormal basis of F. Assume that Alice and Fred preshare the pure bipartite state

$$|\psi_{AF}\rangle := \frac{1}{\sqrt{L}} \sum_{l=1}^{L} |\psi_{A,l}\rangle \otimes |l\rangle_F , \tag{D13}$$

and define the averaged state $\overline{\sigma}_A$ as

$$\begin{aligned}\overline{\sigma}_A &:= \frac{1}{L}\sum_{l=1}^{L} Q_{A,l}\mathcal{G}(|\psi_{A,l}\rangle\langle\psi_{A,l}|)Q_{A,l} \\ &= \frac{1}{L}\sum_{l=1}^{L}\sum_{k\in\mathcal{S}_l}\mathcal{G}(\Pi_k Q_{A,l}|\psi_{A,l}\rangle\langle\psi_{A,l}|Q_{A,l}\Pi_k),\end{aligned} \tag{D14}$$

where the second inequality follows from Lemma 8. We have the following result regarding the one-way LOCC classical communication capability of $|\psi_{AF}\rangle$, which will be shown in step 5 below.

Lemma 14. *Let M be the message size. Let $E \equiv (\mathtt{g}_1, \ldots, \mathtt{g}_M)$ be a random coding such that each codeword $\mathtt{g}_m$ is chosen independently and uniformly from G. We use the typewriter font $\mathtt{g}$ to indicate that it is a random variable. For a chosen encoder $(g_1, \ldots, g_M)$, there exists a one-way LOCC decoder such that the resulting protocol $\mathcal{C}$'s expected decoding error is upper bounded as*

$$\mathbb{E}_E e(\mathcal{C}(E)) \le \frac{8M^s}{(1-\delta)^s}\operatorname{Tr}[\overline{\sigma}_A^{1+s}] + 2\delta \tag{D15}$$

with $s \in (0,1)$ and δ defined as

$$\begin{aligned}\delta &:= 1 - \langle\psi_{AF}|Q_{AF}|\psi_{AF}\rangle = 1 - \frac{1}{L}\sum_{l=1}^{L}\langle\psi_{A,l}|Q_{A,l}|\psi_{A,l}\rangle \\ &= 1 - \frac{1}{L}\sum_{l=1}^{L}\mathcal{G}(|\psi_{A,l}\rangle\langle\psi_{A,l}|),\end{aligned} \tag{D16}$$

where the second equation follows from Eqs. (D12) and (D13), and the final equation follows from the invariant property of $Q_{A,l}$ for the group action.

Step 3. We prove Theorem 6 by applying Lemma 14 to the measurement induced by the two-universal hash function. That is, we adopt the random coding argument and show that the expected decoding error for protocols generated by randomly selecting codewords according to the uniform distribution and measurements according to the two-universal hash function is upper bounded.

In the first stage, Fred performs the projective measurement $\{\Pi^{f,t}\}$, dividing system F into T subspaces. When the outcome is t, the postmeasurement state on AF is

$$|\Psi_{AF}^{f,t}\rangle := \frac{1}{\sqrt{L}}\sum_{n\in f^{-1}(t)} |\psi_{A,n}\rangle |n\rangle_F . \tag{D17}$$

The outcome is communicated to Bob via a classical noiseless channel. Set $\mathbf{Q}_{AF}^{f,t} := \sum_{n\in f^{-1}(t)} Q_{A,n}^{f,t} \otimes |n\rangle\langle n|_F$, which is the conditional version of $\mathbf{Q}^{f,t}$. One can check that

$$\langle\Psi_{AF}^{f,t}|\mathbf{Q}_{AF}^{f,t}|\Psi_{AF}^{f,t}\rangle = \frac{1}{L}\sum_{n\in f^{-1}(t)}\langle\psi_{A,n}|Q_{A,n}^{f,t}|\psi_{A,n}\rangle = 1-\delta^{f,t}. \tag{D18}$$

In the second stage, we apply Lemma 14 to the postmeasurement state $|\Psi_{AF}^{f,t}\rangle$ with corresponding measurement $\mathbf{Q}^{f,t}$ to implement classical communication from Alice to Bob. We can do so because measurement $\mathbf{Q}^{f,t}$ satisfies the prerequisite given in Lemma 14. Consequently, there exists a communication protocol with one-way LOCC decoder $\mathcal{C}(e,f,t)$ depending on an encoder e, a hash function f, and t such that the decoding error $\epsilon(\mathcal{C}(e,f,t))$ satisfies the condition

$$\mathbb{E}_E\epsilon(\mathcal{C}(E,f,t)) \leq \frac{8M^s}{(1-\delta^{f,t})^s}\,\mathrm{Tr}\left[\left(\frac{1}{L}\sum_{n\in f^{-1}(t)} Q_{A,n}^{f,t}\overline{\rho}_{A,n}Q_{A,n}^{f,t}\right)^{1+s}\right] + 2\delta^{f,t}. \tag{D19}$$

Averaging over all possible randomly generated codewords E according to the uniform distribution, a two-universal hash function $\hat{F}$, and the random variable $\hat{T}$, we can upper bound the expected value of the decoding error as

$$\begin{aligned}
&\mathbb{E}_{E,\hat{F},\hat{T}}\epsilon(\mathcal{C}(E,\hat{F},\hat{T}))\\
&= \Pr\left(\delta^{\hat{F},\hat{T}} \geq \frac{1}{2}\right)\mathbb{E}_{E,\hat{F},\hat{T}|\delta^{\hat{F},\hat{T}}\geq 1/2}\epsilon(\mathcal{C}(E,\hat{F},\hat{T}))\\
&\quad + \Pr\left(\delta^{\hat{F},\hat{T}} < \frac{1}{2}\right)\mathbb{E}_{E,\hat{F},\hat{T}|\delta^{\hat{F},\hat{T}}<1/2}\epsilon(\mathcal{C}(E,\hat{F},\hat{T})) \qquad \text{(D20a)}\\
&\overset{\text{(a)}}{\leq} \Pr\left(\delta^{\hat{F},\hat{T}} \geq \frac{1}{2}\right) + \Pr\left(\delta^{\hat{F},\hat{T}} < \frac{1}{2}\right)\mathbb{E}_{E,\hat{F},\hat{T}|\delta^{\hat{F},\hat{T}}<1/2}\epsilon(\mathcal{C}(E,\hat{F},\hat{T}))\\
&\overset{\text{(b)}}{\leq} 2\mathbb{E}_{\hat{F},\hat{T}}\delta^{\hat{F},\hat{T}} + \Pr\left(\delta^{\hat{F},\hat{T}} < \frac{1}{2}\right)\mathbb{E}_{E,\hat{F},\hat{T}|\delta^{\hat{F},\hat{T}}<1/2}\epsilon(\mathcal{C}(E,\hat{F},\hat{T}))\\
&\overset{\text{(c)}}{\leq} 4\mathbb{E}_{\hat{F},\hat{T}}\delta^{\hat{F},\hat{T}}\\
&\quad + \Pr\left(\delta^{\hat{F},\hat{T}} < \frac{1}{2}\right)\mathbb{E}_{\hat{F},\hat{T}|\delta^{\hat{F},\hat{T}}<1/2}\left\{\frac{8M^s}{(1-\delta^{\hat{F},\hat{T}})^s}\,\mathrm{Tr}\left[\left(\frac{1}{L}\sum_{n\in\hat{F}^{-1}(\hat{T})} Q_{A,n}^{\hat{F},\hat{T}}\overline{\rho}_{A,n}Q_{A,n}^{\hat{F},\hat{T}}\right)^{1+s}\right]\right\}\\
&\overset{\text{(d)}}{\leq} 4\mathbb{E}_{\hat{F},\hat{T}}\delta^{\hat{F},\hat{T}} + 2^{s+3}M^s\mathbb{E}_{\hat{F},\hat{T}}\,\mathrm{Tr}\left[\left(\frac{1}{L}\sum_{n\in\hat{F}^{-1}(\hat{T})} Q_{A,n}^{\hat{F},\hat{T}}\overline{\rho}_{A,n}Q_{A,n}^{\hat{F},\hat{T}}\right)^{1+s}\right]\\
&\overset{\text{(e)}}{\leq} 4\mathbb{E}_{\hat{F},\hat{T}}\delta^{\hat{F},\hat{T}} + 2^{s+3}M^s\mathbb{E}_{\hat{F},\hat{T}}\,\mathrm{Tr}\left[\left(\frac{1}{L}\sum_{n\in\hat{F}^{-1}(\hat{T})} \overline{\rho}_{A,n}\right)^{1+s}\right]\\
&\overset{\text{(f)}}{\leq} 4T^{-s'}e^{s'\overline{H}_{1-s'}(F|A)_\xi} + 2^{s+3}\frac{M^s}{L^s}e^{-s\overline{H}_{1+s}(A|F)_\xi} + 2^{s+3}M^s e^{-s\overline{H}_{1+s}(A)_\xi}\\
&= 4e^{-s'[\log T-\overline{H}_{1-s'}(F|A)_\xi]} + 2^{s+3}e^{-s[\overline{H}_{1+s}(A|F)_\xi+\log N-\log T-\log M]} + 2^{s+3}e^{-s[\overline{H}_{1+s}(A)_\xi-\log M]}\\
&\overset{\text{(g)}}{=} 4e^{-s'[\log T-\overline{H}_{1-s'}(F|A)_\xi]} + 2^{s+3}e^{-s[\overline{H}_{1+s}(AF)_\xi-\log T-\log M]} + 2^{s+3}e^{-s[\overline{H}_{1+s}(A)_\xi-\log M]}, \qquad \text{(D20b)}
\end{aligned}$$

where

1. $\mathbb{E}_{X|B}$ expresses the conditional expectation with respect to the variable X conditioned on B,
2. inequality (a) follows from the fact that the decoding error is less than 1,
3. inequality (b) follows from the Markov inequality [149, Eq. (3.31)] that $\Pr(\delta^{\hat{F},\hat{T}} \geq 1/2) \leq 2\mathbb{E}_{\hat{F},\hat{T}}\delta^{\hat{F},\hat{T}}$,

4. inequality (c) follows from Eq. (D19),
5. inequality (d) follows from the relation that $\delta^{\hat{F},\hat{T}} < 1/2$ implies that $(1-\delta^{\hat{F},\hat{T}})^{-s} < 2^s$; note that this relation is the essential reason why we divide the expectation into two regions—$\delta^{\hat{F},\hat{T}} \geq 1/2$ and $\delta^{\hat{F},\hat{T}} < 1/2$—in Eq. (D20a), since otherwise we cannot bound the term $(1-\delta^{\hat{F},\hat{T}})^{-s}$,
6. inequality (e) follows from the fact that the measurement element satisfies $0 \leq Q^{f,t}_{A,n} \leq \mathbb{1}$ and thus the mutual commutativity property guarantees that $Q^{f,t}_{A,n}\overline{\rho}_{A,y}Q^{f,t}_{A,n} \leq \overline{\rho}_{A,n}$,
7. inequality (f) follows from the expectation estimations in Eqs. (D10) and (D11) with respect to the two-universal hash function, and
8. equality (g) follows from the fact that ξ_F is the completely mixed state.

We set $s' = s$ in Eq. (D20b) and solve the equation with respect to the variable $\log T$,

$$\log T - \overline{H}_{1-s}(F|A)_\xi = \overline{H}_{1+s}(AF)_\xi - \log T - \log M, \tag{D21}$$

yielding

$$\log T = \tfrac{1}{2}(\overline{H}_{1-s}(F|A)_\xi + \overline{H}_{1+s}(AF)_\xi - \log M). \tag{D22}$$

Based on these choices, we can conclude from Eq. (D20) that there exists a concrete communication protocol one-way LOCC decoder $\mathcal{C}(e,f,t)$ for carefully chosen encoding e and the two-universal hash function f such that its decoding error is upper bounded for $s \in [0,1]$ as

$$\begin{aligned}\epsilon(\mathcal{C}(e,f,t)) &\leq (4+2^{s+3})e^{-(s/2)[\overline{H}_{1+s}(AF)_\xi - \overline{H}_{1-s}(F|A)_\xi - \log M]} + 2^{s+3}e^{-s[\overline{H}_{1+s}(A)_\xi - \log M]} \\ &\leq (4+16)e^{-(s/2)[\overline{H}_{1+s}(AF)_\xi - \overline{H}_{1-s}(F|A)_\xi - \log M]} + 16e^{-s[\overline{H}_{1+s}(A)_\xi - \log M]}.\end{aligned} \tag{D23}$$

Since $\epsilon(\mathcal{C}(e,f,t)) \leq 1$, we have

$$\begin{aligned}\epsilon(\mathcal{C}(e,f,t)) &\leq 36\min(1, \max(e^{-(s/2)[\overline{H}_{1+s}(AF)_\xi - \overline{H}_{1-s}(F|A)_\xi - \log M]}, e^{-s[\overline{H}_{1+s}(A)_\xi - \log M]})) \\ &\leq 36\min(1, \max(e^{-(s/2)[\overline{H}_{1+s}(AF)_\xi - \overline{H}_{1-s}(F|A)_\xi - \log M]}, e^{-(s/2)[\overline{H}_{1+s}(A)_\xi - \log M]})) \\ &= 36\min(1, e^{-\min((s/2)[\overline{H}_{1+s}(AF)_\xi - \overline{H}_{1-s}(F|A)_\xi - \log M], \frac{s}{2}[\overline{H}_{1+s}(A)_\xi - \log M])}) \\ &= 36\min(1, e^{-(1/2)\mathcal{L}_\xi(-\log M)}).\end{aligned} \tag{D24}$$

That is,

$$-2\log\frac{\epsilon(\mathcal{C}(e,f,t))}{36} \geq \mathcal{L}_\xi(-\log M). \tag{D25}$$

Since $\mathcal{L}$ is monotonically increasing, we have

$$-\mathcal{L}_\xi^{-1}\left(-2\log\frac{\epsilon(\mathcal{C}(e,f,t))}{36}\right) \leq \log M. \tag{D26}$$

Step 4. To show Lemma 14, we derive a key lemma using Lemma 8.

Define the Lth root of unity $\zeta := \exp(2\pi i/L)$. From $\{|l\rangle_F\}_{l=1}^L$ we construct the induced Fourier basis measurement $\{|\mathbf{b}_F^{l'}\rangle\}_{l'=1}^L$ via

$$|\mathbf{b}_F^{l'}\rangle := \frac{1}{\sqrt{L}}\sum_{l=1}^{L}\zeta^{ll'}\,|l\rangle_F, \qquad l' = 1,\ldots,L. \tag{D27}$$

For each $l' = 1,\ldots,L$, defined the subnormalized pure quantum state

$$|\phi_{A,l'}\rangle := \langle \mathbf{b}_F^{l'}|Q_{AF}|\psi_{AF}\rangle = \frac{1}{L}\sum_{l=1}^{L}\zeta^{-ll'}Q_{A,l}|\psi_{A,l}\rangle, \tag{D28}$$

whose norm can be calculated as

$$\langle\phi_{A,l}|\phi_{A,l}\rangle = \frac{1}{L^2}\sum_{l=1}^{L}\langle\psi_{A,l}|Q_{A,l}|\psi_{A,l}\rangle = \frac{1-\delta}{L}, \tag{D29}$$

where the last equality follows from the definition of δ (D16).

Fred now performs this Fourier basis measurement on $|\psi_{AF}\rangle$. After measurement, Fred holds the classical outcome l and Alice holds the postmeasurement pure state. This leads to the classical-quantum state

$$\sigma_{AF} := \sum_{l=1}^{L}\langle \mathbf{b}_F^l|\psi\rangle_{AF}\langle\psi|\mathbf{b}_F^l\rangle \otimes |l\rangle\langle l|_F. \tag{D30}$$

Applying the pinching lemma (cf. Lemma 10) to the quantum state $|\psi_{AF}\rangle$ and the binary projective measurement $\{Q_{AF}, \mathbb{1}_{AF} - Q_{AF}\}$ gives

$$|\psi\rangle\langle\psi|_{AF} \le 2Q_{AF}|\psi\rangle\langle\psi|_{AF}Q_{AF} + 2(\mathbb{1}_{AF} - Q_{AF})|\psi\rangle\langle\psi|_{AF}(\mathbb{1}_{AF} - Q_{AF}). \tag{D31}$$

Substituting Eq. (D31) into Eq. (D30) yields the following inequality regarding σ_{AF}:

$$\begin{aligned}\sigma_{AF} &\le \sum_{l=1}^{L}\langle \mathbf{b}_F^l|(2Q|\psi\rangle\langle\psi|Q + 2(\mathbb{1}-Q)|\psi\rangle\langle\psi|(\mathbb{1}-Q))|\mathbf{b}_F^l\rangle \otimes |l\rangle\langle l|_F \\ &= 2\sum_{l=1}^{L}|\phi_{A,l}\rangle\langle\phi_{A,l}| \otimes |l\rangle\langle l|_F + 2\sum_{l=1}^{L}\langle \mathbf{b}_F^l|(\mathbb{1}-Q_{AF})|\psi\rangle\langle\psi|(\mathbb{1}-Q_{AF})|\mathbf{b}_F^l\rangle \otimes |l\rangle\langle l|_F.\end{aligned} \tag{D32}$$

From the unitary representation U of a group G, for each $g \in G$, we define the quantum state

$$g \mapsto W_{BF}^g := \frac{1}{1-\delta}\sum_{l=1}^{L}U_g|\phi_{A,l}\rangle\langle\phi_{A,l}|U_g^\dagger \otimes |l\rangle\langle l|_F, \tag{D33}$$

where U_g is the unitary operator corresponding to g and $\phi_{A,l}$ is defined in Eq. (D28). In fact, relation (D29) guarantees that W_{BF}^g is a quantum state for each $g \in G$. Then, we have the following lemma.

Lemma 15. *The set of quantum states $\{W_{BF}^g\}_{g\in G}$ has the following averaged state with respect to the Haar measure ν:*

$$\overline{W}_{BF} := \int_G W_{BF}^g \nu(dg) = \frac{1}{1-\delta}\overline{\sigma}_A \otimes \pi_F. \tag{D34}$$

Proof. Using Lemma 8, we show Eq. (D34) as follows:

$$\begin{aligned}
\overline{W}_{BF} &:= \int_G W_{BF}^g \nu(dg) \\
&= \int_G \left(\sum_l \left(\frac{1}{1-\delta} U_g |\phi_{A,l}\rangle\langle\phi_{A,l}| U_g^\dagger \right) \otimes |l\rangle\langle l|_F \right) \nu(dg) \\
&\overset{(a)}{=} \sum_{l=1}^{L} \frac{1}{1-\delta} \mathcal{G}(\phi_{A,l}) \otimes |l\rangle\langle l|_F \\
&\overset{(b)}{=} \sum_{l=1}^{L} \frac{1}{1-\delta} \sum_k \mathcal{G}(\Pi_k \phi_{A,l}) \otimes |l\rangle\langle l|_F \\
&= \sum_{l=1}^{L} \frac{1}{(1-\delta)L^2} \sum_{l'=1}^{L} \sum_k \mathcal{G}(\Pi_k Q_{A,l'} \psi_{A,l'}) \otimes |l\rangle\langle l|_F \\
&= \frac{1}{(1-\delta)L^2} \sum_{l'=1}^{L} \sum_k \mathcal{G}(\Pi_k Q_{A,l'} \psi_{A,l'}) \otimes \left(\sum_{l=1}^{L} \frac{1}{L} |l\rangle\langle l|_F \right) \\
&\overset{(c)}{=} \frac{1}{1-\delta} \overline{\sigma}_A \otimes \pi_F.
\end{aligned} \tag{D35}$$

Equality (a) follows from the definition of $\mathcal{G}$, equality (b) follows from Lemma 8, and equality (c) follows from Eq. (D14). ■

Step 5. Now we show Lemma 14 using Lemma 15. Applying the direct part of the classical-quantum channel coding theorem, i.e., Lemma 12, we conclude that there exist an encoder $(g_1, \ldots, g_M)$ and a decoder $\mathbf{\Gamma} \equiv \{\Gamma^m\}_{m=1}^M$ in BF as a one-way LOCC measurement from Fred to Bob such that the decoding error is upper bounded for arbitrary $s \in [0,1]$:

$$\begin{aligned}
\text{decoding error} &:= \frac{1}{M} \sum_{m=1}^{M} \mathrm{Tr}[W^{g_m}(\mathbb{1} - \Gamma^m)] \\
&\overset{(a)}{\leq} 4M^s \int_G \mathrm{Tr}[(W^g)^{1-s} \overline{W}^s] \nu(dg) \\
&= 4M^s \int_G \mathrm{Tr}\left[\left(\sum_{l=1}^{L} \frac{L}{1-\delta} U_g |\phi_{A,l}\rangle\langle\phi_{A,l}| U_g^\dagger \otimes \frac{1}{L} |l\rangle\langle l|_F \right)^{1-s} \right. \\
&\qquad \left. \times \left(\frac{1}{1-\delta} \overline{\sigma}_A \otimes \pi_F \right)^s \right] \nu(dg) \\
&= 4M^s \int_G \mathrm{Tr}\left[\sum_l \left(\frac{L}{1-\delta} U_g |\psi_{A,l}\rangle\langle\psi_{A,l}| U_g^\dagger \right)^{1-s} \left(\frac{1}{1-\delta} \overline{\sigma}_A \right)^s \otimes \frac{1}{L} |l\rangle\langle l|_F \right] \nu(dg) \\
&\overset{(b)}{=} 4M^s \int_G \mathrm{Tr}\left[\sum_l \left(\frac{L}{1-\delta} U_g |\phi_{A,l}\rangle\langle\phi_{A,l}| U_g^\dagger \right) \left(\frac{1}{1-\delta} \overline{\sigma}_A \right)^s \otimes \frac{1}{L} |l\rangle\langle l|_F \right] \nu(dg) \\
&= 4M^s\, \mathrm{Tr}\left[\int_G \left(\sum_l \left(\frac{1}{1-\delta} U_g |\phi_{A,l}\rangle\langle\phi_{A,l}| U_g^\dagger \right) \otimes |l\rangle\langle l|_F \right) \right. \\
&\qquad \left. \times \left(\left(\frac{1}{1-\delta} \overline{\sigma}_A \right)^s \otimes I_F \right) \nu(dg) \right]
\end{aligned}$$

$$\begin{aligned}
&\overset{\text{(c)}}{=} \frac{4M^s}{(1-\delta)} \operatorname{Tr}\left[\left(\sum_l \overline{\sigma}_A \otimes \frac{1}{L}|l\rangle\langle l|_F\right)\left(\left(\frac{1}{1-\delta}\overline{\sigma}_A\right)^s \otimes I_F\right)\right] \\
&= \frac{4M^s}{(1-\delta)^{1+s}} \operatorname{Tr}\left[\sum_l \overline{\sigma}_A^{1+s} \otimes \frac{1}{L}|l\rangle\langle l|_F\right] \\
&= \frac{4M^s}{(1-\delta)^{1+s}} \operatorname{Tr}[\overline{\sigma}_A^{1+s}]. \qquad \text{(D36a)}
\end{aligned}$$

Inequality (a) follows from Lemma 12, equality (b) follows from Eq. (D29), implying that $[L/(1-\delta)]U_g|\phi\rangle\langle\phi|_{A,l}U_g^\dagger$ is a normalized pure state, and equality (c) follows from Lemma 15.

Now we are ready to give a concrete protocol $\mathcal{C}$ achieving the decoding error concluded in Eq. (D15). In this protocol, Fred adopts the Fourier basis measurement $\{|\mathbf{b}_F^l\rangle\}_{l=1}^L$, Alice adopts the encoding $(g_1,\ldots,g_M)$, and Bob adopts the decoder $\mathbf{\Gamma}$ originally designed for the classical-quantum channel $g \mapsto W_{AF}^g$. This is a communication protocol with one-way LOCC decoder for $|\psi_{AF}\rangle$ since $\mathbf{\Gamma}$ is essentially a one-way LOCC decoder from Fred to Bob. Thanks to the above analysis, we can evaluate the decoding error of $\mathcal{C}$ as

$$\begin{aligned}
e(\mathcal{C}) &:= \frac{1}{M}\sum_{m=1}^M \operatorname{Tr}[(U_{g_m}\sigma_{AF}U_{g_m}^\dagger)(\mathbb{1}-\Gamma^m)] \\
&\overset{\text{(a)}}{\leq} \frac{1}{M}\sum_{m=1}^M \operatorname{Tr}\left[U_{g_m}\left(2\sum_{l=1}^L |\phi_{A,l}\rangle\langle\phi_{A,l}| \otimes |l\rangle\langle l|_F\right)U_{g_m}^\dagger(\mathbb{1}-\Gamma^m)\right] \\
&\quad + \frac{1}{M}\sum_{m=1}^M \operatorname{Tr}\left[U_{g_m}\left(2\sum_{l=1}^L \langle\mathbf{b}^l|(\mathbb{1}-Q)|\psi\rangle\langle\psi|(\mathbb{1}-Q)|\mathbf{b}^l\rangle \otimes |l\rangle\langle l|_F\right)U_{g_m}^\dagger(\mathbb{1}-\Gamma^m)\right] \\
&\overset{\text{(b)}}{=} \frac{2(1-\delta)}{M}\sum_{m=1}^M \operatorname{Tr}[W^{g_m}(\mathbb{1}-\Gamma^m)] \\
&\quad + \frac{1}{M}\sum_{m=1}^M \operatorname{Tr}\left[U_{g_m}\left(2\sum_{l=1}^L \langle\mathbf{b}^l|(\mathbb{1}-Q)|\psi\rangle\langle\psi|(\mathbb{1}-Q)|\mathbf{b}^l\rangle \otimes |l\rangle\langle l|_F\right)U_{g_m}^\dagger(\mathbb{1}-\Gamma^m)\right] \\
&\overset{\text{(c)}}{\leq} \frac{8M^s}{(1-\delta)^s}\operatorname{Tr}[\overline{\sigma}_A^{1+s}] + \frac{2}{M}\sum_{m=1}^M \operatorname{Tr}\left[\sum_{l=1}^L \langle\mathbf{b}^l|(\mathbb{1}-Q)|\psi\rangle\langle\psi|(\mathbb{1}-Q)|\mathbf{b}^l\rangle \otimes |l\rangle\langle l|_F\right] \\
&\overset{\text{(d)}}{\leq} \frac{8M^s}{(1-\delta)^s}\operatorname{Tr}[\overline{\sigma}_A^{1+s}] + 2\delta, \qquad \text{(D37a)}
\end{aligned}$$

where (a) follows from Eq. (D32), (b) follows from the definition of W_{BF}^g (D33), (c) follows from Eq. (D36) and $\Gamma^m \geq 0$, and (d) follows from the fact that $\{|\mathbf{b}_F^l\rangle\}$ forms an orthonormal basis of F and the definition of δ (D16). This concludes the proof of Theorem 6. ∎

Remark 2: Actually, our achievability proof (Theorem 6 and Lemma 14) is inspired by the proof of Theorem 1 of Smolin *et al.* [121], which we refer to as SVW. In the following, we compare in detail the similarity and uniqueness between our proof and SVW. In general, both proofs are composed of two parts. In the first part, we apply a surjective linear hash function. This mimics choosing the typical subspaces in SVW. In the second part, Fred measures on a Fourier basis. This is the one-shot correspondence to the Fourier basis measurement in SVW. However, our task is different from the task that is considered in SVW. We need to invent different operations for both the sender and the receiver and manage a different evaluation method for the decoding error probability. On the other hand, SVW does not assume uniform distribution on the codewords *a priori*. However, we do have this assumption due to the special structure of the task under consideration. This uniformity assumption renders a more complicated proof so that it becomes more difficult to derive an exponential upper bound.

b. Asymptotic direct part

Based on the one-shot direct part in Theorem 6, we can show the following coding theorem, which concludes the second inequality of Eq. (D1).

Theorem 7 (Direct part)**.** *Let $|\Psi\rangle_{AF}$ be a bipartite pure quantum state. When the (projective) unitary representation U on $\mathcal{H}_A$ satisfies Assumption 1 (the multiplicity-free condition), it holds that*

$$H(\mathcal{G}(\Psi_A)) \leq C_{\mathfrak{E}_g,\mathfrak{D}_\rightarrow}(\Psi_{AF}), \tag{D38}$$

where $\mathcal{G}$ is the G-twirling operation defined in Eq. (18), $\Psi_A = \mathrm{Tr}_F\, \Psi_{AF}$, and $C_{\mathfrak{E}_g,\mathfrak{D}_\rightarrow}(\Psi_{AF})$ is defined in Eq. (32).

Proof. We focus on the functions $-s\overline{H}_{1+s}(AF)_\xi$, $s\overline{H}_{1-s}(F|A)_\xi$, and $-s\overline{H}_{1+s}(A)_\xi$, which are convex functions for s. When s is close to 0, using Taylor expansions with respect to s, they are approximated to $-sH(AF)_\xi + s^2V(AF)_\xi/2$, $sH(F|A)_\xi + s^2V(F|A)_\xi/2$, and $-sH(A)_\xi + s^2V(A)_\xi/2$, respectively. Hence,

$$\min(s\overline{H}_{1+s}(AF)_\xi - s\overline{H}_{1-s}(F|A)_\xi, s\overline{H}_{1+s}(A)_\xi) \tag{D39}$$

is approximated as $sH(A)_\xi - s^2V_\xi/2$, where $V_\xi := \max(V(A)_\xi + V(AF)_\xi, V(F|A)_\xi)$. Thus, we have

$$\begin{aligned} & s(-nH(A)_\xi + \sqrt{n}r) + \min(s\overline{H}_{1+s}(AF)_{\xi^{\otimes n}} - s\overline{H}_{1-s}(F|A)_{\xi^{\otimes n}}, s\overline{H}_{1+s}(A)_{\xi^{\otimes n}}) \\ &= s(-nH(A)_\xi + \sqrt{n}r) + snH(A)_\xi - \frac{ns^2}{2}V_\xi + o(ns^2) \\ &= \sqrt{n}sr - \frac{ns^2}{2}V_\xi + o(ns^2) \\ &= -\frac{n}{2}V_\xi\left(s - \frac{r}{\sqrt{n}V_\xi}\right)^2 + \frac{r^2}{2V_\xi} + o(ns^2). \end{aligned} \tag{D40}$$

Since the maximum of the above value for s is realized around $s = r/\sqrt{n}V_\xi$, we have

$$\lim_{n\to\infty} \mathcal{L}_{\xi^{\otimes n}}(-nH(A)_\xi + \sqrt{n}r) = \frac{r^2}{2V_\xi}, \tag{D41}$$

which implies that

$$-\mathcal{L}^{-1}_{\xi^{\otimes n}}(-\log\epsilon) = nH(A)_\xi - \sqrt{-2nV_\xi\log\epsilon} + o(\sqrt{n}). \tag{D42}$$

Combining the above result with Theorem 6 yields Eq. (D38). ■

2. Strong converse part

In the strong converse part, we show that $H(\mathcal{G}(\Psi_A))$ is a strong converse bound for all the quantities mentioned in Eq. (43) regardless of Assumption 1 (the multiplicity-free condition). This concludes the first inequality of Eq. (D1).

Theorem 8 (Strong converse part)**.** *Let $|\Psi\rangle_{AF}$ be a bipartite pure quantum state. It holds that*

$$C^\dagger_{\mathfrak{E}_{ppt},\mathfrak{D}_{ppt}}(\Psi_{AF}) \leq H(\mathcal{G}(\Psi_A)), \tag{D43}$$

$$C^\dagger_{\mathfrak{E}_p,\mathfrak{D}_{sep}}(\Psi_{AF}) \leq H(\mathcal{G}(\Psi_A)), \tag{D44}$$

where $\mathcal{G}$ is the G-twirling operation defined in Eq. (18), $\Psi_A = \mathrm{Tr}_F\,\Psi_{AF}$, and $C^\dagger_{\mathfrak{E},\mathfrak{D}}$ is defined in Eq. (34).

To show Theorem 8, we prepare the following lemma.

Lemma 16. *For any $\epsilon > 0$, we choose a sufficiently large integer N such that any $n \geq N$ satisfies the following two conditions.*

(i) There exists a projection P_A such that

$$[P_A, U_g] = 0 \quad \text{for } g \in G^n, \tag{D45}$$

$$[P_A, \mathcal{G}(\Psi_A)^{\otimes n}] = 0, \tag{D46}$$

$$\operatorname{Tr} P_A \leq e^{n(H(\mathcal{G}(\Psi_A))+\epsilon)}, \tag{D47}$$

$$\operatorname{Tr}(I - P_A)\mathcal{G}(\Psi_A)^{\otimes n} \leq \epsilon, \tag{D48}$$

$$\mathcal{G}(P_A) = P_A. \tag{D49}$$

(ii) There exists a projection P_F such that

$$[P_F, \Psi_F^{\otimes n}] = 0, \tag{D50}$$

$$\operatorname{Tr} P_F \leq e^{n(H(\Psi_A)+\epsilon)}, \tag{D51}$$

$$\|P_F \Psi_F^{\otimes n} P_F\| \leq e^{-n(H(\Psi_A)-\epsilon)}, \tag{D52}$$

$$\operatorname{Tr}(I - P_F)\Psi_F^{\otimes n} \leq \epsilon. \tag{D53}$$

Proof. In this proof, we employ the following notation. Given a Hermitian matrix X with the diagonalization $\sum_j x_j |u_j\rangle\langle u_j|$, we define the projection $\{X \geq R\}$ with a constant R as $\sum_{j:x_j \geq R} |u_j\rangle\langle u_j|$. We also define the projection $\{R' \geq X \geq R\}$ with constants R, R' in the same way.

The state $\mathcal{G}(\Psi_A)$ is written as $\sum_k P_K(k)\pi_k$, where π_k is the completely mixed state on $\mathcal{H}_k$. Also, the reduced density Ψ_F is written as $\sum_f P_F(f)|u_f\rangle\langle u_f|$. Then, we have

$$\sum_k P_K(k)(\log d_k - \log P_K(k)) = H(\mathcal{G}(\Psi_A)), \tag{D54}$$

$$-\sum_f P_F(f) \log P_F(f) = H(\Psi_F). \tag{D55}$$

Because of the law of large numbers, for any $\epsilon > 0$, we choose a sufficiently large integer N such that any $n \geq N$ satisfies the conditions

$$P_K^n\left\{(k_1, \ldots, k_n) \;\middle|\; \sum_{i=1}^n \log d_{k_i} - \log P_K(k_i) \leq nR_1\right\} \geq 1 - \epsilon, \tag{D56}$$

$$P_F^n\left\{(f_1, \ldots, f_n) \;\middle|\; nR_3 \leq -\sum_{i=1}^n \log P_F(f_i) \leq nR_2\right\} \geq 1 - \epsilon, \tag{D57}$$

where $R_1 := H(\mathcal{G}(\Psi_A)) + \epsilon$, $R_2 := H(\Psi_F) + \epsilon$, and $R_3 := H(\Psi_F) - \epsilon$.

Then, we choose the projections P_A and P_F as

$$P_A := \{\mathcal{G}(\Psi_A)^{\otimes n} \geq 2^{-nR_1}\}, \tag{D58}$$

$$P_F := \{2^{-nR_3} \geq \Psi_F^{\otimes n} \geq 2^{-nR_2}\}. \tag{D59}$$

The projection satisfies Eq. (D46). Since $\mathcal{G}(\Psi_A)^{\otimes n}$ is commutative with U_g for $g \in G^n$, we have Eq. (D45). Since $\mathcal{G}(\Psi_A)^{\otimes n}$ is a constant on each irreducible space, we have Eq. (D50). Condition (D56) implies condition (D48). Since

$$\operatorname{Tr} P_A 2^{-nR_1} \leq \operatorname{Tr} P_A \mathcal{G}(\Psi_A)^{\otimes n} \leq 1, \tag{D60}$$

the definition $R_1 := H(\mathcal{G}(\Psi_A)) + \epsilon$ implies condition (D47). In the same way, we have conditions (D50), (D51), and (D53). The definitions of P_F and R_3 imply condition (D52). ■

Proof. The essential tool to prove Eqs. (D43) and (D44) is the following inequality, which is shown in Ref. [103, Eq. (8.217)]. We denote the transpose operation on F by τ_F. For any bipartite positive semidefinite rank-one operator X on $\mathcal{H}_{AF}$, we have the relation

$$|\tau_F(X)| = \sqrt{\operatorname{Tr}_F X} \otimes \sqrt{\operatorname{Tr}_A X}. \tag{D61}$$

In fact, Hayashi [103, Eq. (8.217)] showed Eq. (D61) by using the transpose on a specific basis. While the map τ_F depends on the choice of the basis, $|\tau_F(X)|$ does not depend on it. Consider the map $X \mapsto U^\dagger \tau_F(UXU^\dagger)U$ by using a unitary on $\mathcal{H}_F$. Then, we have

$$\begin{aligned}
|U^\dagger \tau_F(UXU^\dagger)U|^2 &= U^\dagger \tau_F(UXU^\dagger)UU^\dagger \tau_F(UXU^\dagger)U \\
&= U^\dagger \tau_F(UXU^\dagger)\tau_F(UXU^\dagger)U \\
&= U^\dagger |\tau_F(UXU^\dagger)|^2 U \\
&= U^\dagger \sqrt{\operatorname{Tr}_F X} \otimes \sqrt{\operatorname{Tr}_A UXU^\dagger} U \\
&= \sqrt{\operatorname{Tr}_F X} \otimes (U^\dagger \sqrt{U(\operatorname{Tr}_A X)U^\dagger} U) \\
&= \sqrt{\operatorname{Tr}_F X} \otimes \sqrt{\operatorname{Tr}_A X}.
\end{aligned} \tag{D62}$$

Hence, $|\tau_F(X)|$ does not depend on the choice of basis.

We now show Eq. (D43) using Lemma 16. That is, for any $\epsilon > 0$, we choose a sufficiently large integer N such that any $n \geq N$ satisfies the (i) and (ii). Conditions (D45) and (D48) imply that

$$\mathrm{Tr}(I - P_A)\Psi_A^{\otimes n} \leq \epsilon. \tag{D63}$$

Hence, $[P_A, P_F] = 0$ and

$$\mathrm{Tr}(I - P_A \otimes P_F)\Psi_{AF}^{\otimes n} \leq 2\epsilon. \tag{D64}$$

Let $\mathcal{C} = (\{\mathcal{E}^m\}, \{\Gamma^m\}) \in (\mathfrak{E}_{\text{ppt}}, \mathfrak{D}_{\text{ppt}})$ be a code for the state $\Psi_{AF}^{\otimes n}$. Since Γ^m is a PPT operator and $\mathcal{E}^m \in \mathfrak{E}_{\text{ppt}}$, $(\mathcal{E}^m)^*(\Gamma^m)$ is also a PPT operator, i.e.,

$$\tau_F((\mathcal{E}^m)^*(\Gamma^m)) \geq 0. \tag{D65}$$

By applying Eq. (D61) to $(I \otimes P_F)\Psi_{AF}^{\otimes n}(I \otimes P_F)$, evaluation (D52) guarantees that

$$\begin{aligned}
&\|\tau_F((P_A P_F)\Psi_{AF}^{\otimes n}(P_A P_F))\| \\
&= \|P_A \otimes I \tau_F((I \otimes P_F)\Psi_{AF}^{\otimes n}(I \otimes P_F))P_A \otimes I\| \\
&\leq \|\tau_F((I \otimes P_F)\Psi_{AF}^{\otimes n}(I \otimes P_F))\| \\
&\leq e^{-n(H(\Psi_A)-\epsilon)}.
\end{aligned} \tag{D66}$$

Hence, we have

$$|\tau_F((P_A \otimes P_F)\Psi_{AF}^{\otimes n}(P_A \otimes P_F))| \leq e^{-n(H(\Psi_A)-\epsilon)} P_A \otimes P_F. \tag{D67}$$

Then, we have

$$\begin{aligned}
&\mathrm{Tr}\,\Gamma^m \mathcal{E}^m((P_A \otimes P_F)\Psi_{AF}^{\otimes n}(P_A \otimes P_F)) \\
&= \mathrm{Tr}(\mathcal{E}^m)^*(\Gamma^m)(P_A \otimes P_F)\Psi_{AF}^{\otimes n}(P_A \otimes P_F) \\
&= \mathrm{Tr}\,\tau_F((\mathcal{E}^m)^*(\Gamma^m))\tau_F((P_A \otimes P_F)\Psi_{AF}^{\otimes n}(P_A \otimes P_F)) \\
&\overset{(a)}{\leq} \mathrm{Tr}\,\tau_F((\mathcal{E}^m)^*(\Gamma^m))|\tau_F((P_A \otimes P_F)\Psi_{AF}^{\otimes n}(P_A \otimes P_F))| \\
&\overset{(b)}{\leq} \mathrm{Tr}\,\tau_F((\mathcal{E}^m)^*(\Gamma^m))e^{-n(H(\Psi_A)-\epsilon)} P_A \otimes P_F \\
&\overset{(c)}{=} \mathrm{Tr}(\mathcal{E}^m)^*(\tau_F(\Gamma^m))\mathcal{G}(e^{-n(H(\Psi_A)-\epsilon)} P_A \otimes P_F) \\
&= \mathrm{Tr}\,\tau_F(\Gamma^m)\mathcal{E}^m(\mathcal{G}(e^{-n(H(\Psi_A)-\epsilon)} P_A \otimes P_F)) \\
&= \mathrm{Tr}\,\tau_F(\Gamma^m)\mathcal{G}(e^{-n(H(\Psi_A)-\epsilon)} P_A \otimes P_F) \\
&= \mathrm{Tr}\,\tau_F(\Gamma^m)e^{-n(H(\Psi_A)-\epsilon)} P_A \otimes P_F \\
&= e^{-n(H(\Psi_A)-\epsilon)}\,\mathrm{Tr}\,\Gamma^m P_A \otimes P_F,
\end{aligned} \tag{D68}$$

where (a), (b), and (c) follow from Eqs. (D65), (D67) and (D49), respectively. Hence, we have

$$\begin{aligned}
s(\mathcal{C}) &= \frac{1}{|\mathcal{M}|}\sum_m \mathrm{Tr}\,\Gamma^m \mathcal{E}^m(\Psi_{AF}^{\otimes n}) \\
&\overset{(a)}{\leq} \frac{1}{|\mathcal{M}|}\sum_m \mathrm{Tr}\,\Gamma^m \mathcal{E}^m((P_A \otimes P_F)\Psi_{AF}^{\otimes n}(P_A \otimes P_F)) + 2\epsilon \\
&\overset{(b)}{\leq} \frac{1}{|\mathcal{M}|}e^{-n(H(\Psi_A)-\epsilon)}\sum_m \mathrm{Tr}\,\Gamma^m P_A \otimes P_F + 2\epsilon \\
&= \frac{1}{|\mathcal{M}|}e^{-n(H(\Psi_A)-\epsilon)}\,\mathrm{Tr}\,P_A \otimes P_F + 2\epsilon \\
&\overset{(c)}{\leq} \frac{1}{|\mathcal{M}|}e^{-n(H(\Psi_A)-\epsilon)}e^{n(H(\mathcal{G}(\Psi_A))+\epsilon)}e^{n(H(\Psi_A)+\epsilon)} + 2\epsilon \\
&\leq \frac{1}{|\mathcal{M}|}e^{n(H(\mathcal{G}(\Psi_A))+3\epsilon)} + 2\epsilon,
\end{aligned} \tag{D69}$$

where (a), (b), and (c) follow from Eqs. (D48), (D53); (D68); and (D47), (D51), respectively. Thus,

$$|\mathcal{M}| \leq \frac{1}{s(\mathcal{C}) - 2\epsilon}e^{n(H(\mathcal{G}(\Psi_A))+3\epsilon)}, \tag{D70}$$

which implies Eq. (D43).

Next, we show Eq. (D44). Let $\mathcal{C} = (\{\mathcal{E}^m\}, \{\Gamma^m\}) \in (\mathfrak{E}_p, \mathfrak{D}_{\text{sep}})$ be a code for the state $\Psi_{AF}^{\otimes n}$. Since Γ^m is a separable operator and $\mathcal{E}^m \in \mathfrak{E}_p$, $(\mathcal{E}^m)^*(\Gamma^m)$ is also a separable operator. Hence, $(\mathcal{E}^m)^*(\Gamma^m)$ is a PPT operator, i.e., we have Eq. (D65). Therefore, in the same way, we can show Eq. (D70), which implies Eq. (D44). ■

APPENDIX E: PROOF OF THEOREM 3

Proof. Applying the channel coding theorem to the classical-quantum channel $g \mapsto U_g \Psi_A U_g^\dagger$ [103], we can easily derive the capacity formula

$$C_{\mathfrak{E}_g,\mathfrak{D}_{\leftrightarrow}}(\Psi_{AF}) = D(\Psi_A \| \mathcal{G}(\Psi_A)). \tag{E1}$$

Hence, to show Eq. (52), it is sufficient to show the strong converse part

$$C^\dagger_{\mathfrak{E}_p,\mathfrak{D}_{\leftrightarrow}}(\Psi_{AF}) \leq D(\Psi_A \| \mathcal{G}(\Psi_A)). \tag{E2}$$

In almost the same way as Eq. (45c), the strong converse argument (E2) can be shown by invoking the metaconverse technique originally invented in Ref. [141] and further investigated in Ref. [142, Chapter 3]. In Proposition 17 below (which will be proved shortly), we upper bound the success probability of any one-shot code $\mathcal{C} \in (\mathfrak{E}_p, \mathfrak{D}_{\leftrightarrow})$ in terms of the sandwiched quantum Rényi entropy, then the strong converse bound follows by block coding. Note that

$\lim_{\alpha\to 1}\widetilde{D}_\alpha(\Psi_A\|\mathcal{G}(\Psi_A)) = D(\Psi_A\|\mathcal{G}(\Psi_A))$. What is more, $\widetilde{D}_\alpha$ is continuous and monotonically decreasing in α. Applying the standard argument outlined in Refs. [141, 143], it follows from Proposition 13 that $D(\Psi_A\|\mathcal{G}(\Psi_A))$ is actually a strong converse bound. ■

Proposition 17. *Any dense coding code* $\mathcal{C}\in(\mathfrak{E}_p,\mathfrak{D}_{\leftrightarrow})$ *obeys the following bound for arbitrary* $\alpha\in(1,\infty)$*:*

$$s(\mathcal{C}) \le \exp\left\{\frac{\alpha-1}{\alpha}(\widetilde{D}_\alpha(\Psi_A\|\mathcal{G}(\Psi_A)) - \log|\mathcal{C}|)\right\} \quad \text{(E3)}$$

with $\widetilde{D}_\alpha$ *the sandwiched quantum Rényi entropy defined in Eq. (8).*

Proof. Given a code $\mathcal{C}=(\{\mathcal{E}^m\}_m,\{\Gamma^{\widehat{m}}\}_{\widehat{m}})\in(\mathfrak{E}_p,\mathfrak{D}_{\leftrightarrow})$, we define the two quantum states

$$\rho_{MA} := \frac{1}{|\mathcal{M}|}\sum_m |m\rangle\langle m|_M\otimes\mathcal{E}^m(\Psi_A), \quad \text{(E4)}$$

$$\sigma_{MA} := \pi_M\otimes\mathcal{G}(\Psi_A), \quad \text{(E5)}$$

where ξ_{MXA} serves as a *test state*. The positive operator

$$T := \sum_m |m\rangle\langle m|_M\otimes\Gamma^m \quad \text{(E6)}$$

satisfies

$$\mathrm{Tr}\,T\rho_{MA} = \frac{1}{|\mathcal{M}|}\sum_m p_{\widehat{M}M}(m|m) = s(\mathcal{C}), \quad \text{(E7)}$$

$$\mathrm{Tr}\,T\sigma_{MA} = \frac{1}{|\mathcal{M}|}\mathrm{Tr}\left[\sum_m\Gamma^m\mathcal{G}(\Psi_A)\right] \overset{(a)}{=} \frac{1}{|\mathcal{M}|}\mathrm{Tr}[\mathcal{G}(\Psi_A)] = \frac{1}{|\mathcal{M}|}, \quad \text{(E8)}$$

where (a) follows from the fact that $\{\Gamma^m\}_m$ is a quantum measurement. Applying the data processing inequality of $\overline{D}_\alpha$ to the binary measurement $\{T, I-T\}$, we have

$$s(\mathcal{C})^\alpha\left(\frac{1}{|\mathcal{C}|}\right)^{1-\alpha} + (1-s(\mathcal{C}))^\alpha\left(1-\frac{1}{|\mathcal{C}|}\right)^{1-\alpha} \le e^{(\alpha-1)\overline{D}_\alpha(\rho_{MA}\|\sigma_{MA})}. \quad \text{(E9)}$$

Thus, we have

$$\begin{aligned} s(\mathcal{C})^\alpha|\mathcal{C}|^{\alpha-1} &= s(\mathcal{C})^\alpha\left(\frac{1}{|\mathcal{C}|}\right)^{1-\alpha} \le s(\mathcal{C})^\alpha\left(\frac{1}{|\mathcal{C}|}\right)^{1-\alpha} \\ &\quad + (1-s(\mathcal{C}))^\alpha\left(1-\frac{1}{|\mathcal{C}|}\right)^{1-\alpha} \\ &\le e^{(\alpha-1)\overline{D}_\alpha(\rho_{MA}\|\sigma_{MA})} \\ &= \frac{1}{|\mathcal{M}|}\sum_m e^{(\alpha-1)\widetilde{D}_\alpha(\mathcal{E}^m(\Psi_A)\|\mathcal{G}(\Psi_A))} \\ &\overset{(a)}{=} \frac{1}{|\mathcal{M}|}\sum_m e^{(\alpha-1)\widetilde{D}_\alpha(\mathcal{E}^m(\Psi_A)\|\mathcal{E}^m\circ\mathcal{G}(\Psi_A))} \\ &\overset{(b)}{\le} \frac{1}{|\mathcal{M}|}\sum_m e^{(\alpha-1)\widetilde{D}_\alpha(\Psi_A\|\mathcal{G}(\Psi_A))}, \end{aligned} \quad \text{(E10)}$$

where (a) follows from the condition $\mathcal{E}^m\circ\mathcal{G}=\mathcal{G}$ for arbitrary $\mathcal{E}^m\in\mathfrak{E}_p$ [cf. the definition in Eq. (37)] and (b) follows from the data processing inequality of the sandwiched quantum Rényi entropy for positive maps [139, Theorem 2]. ■

APPENDIX F: PROOFS OF THEOREMS 4 AND 5

Proof. Note that Eq. (72) can be shown in the same way as Eq. (45c).

Now we show Eq. (73). For a state ρ_{AF}, we have a trace-preserving positive operation $\mathcal{E}_F$ on $\mathcal{H}_F$ such that $\rho_{AF}=\mathcal{E}_F(\Psi_{AF})$, where Ψ_{AF} is a purification of ρ_A. Let $\mathcal{C}=(\{\mathcal{E}^m\},\{\Gamma^m\})\in(\mathfrak{E}_p,\mathfrak{D}_{\mathrm{sep}})$ be a code for state $\rho_{AF}^{\otimes n}$. Since $\{\mathcal{E}_F^*(\Gamma^m)\}$ is a separable measurement, where $\mathcal{E}_F^*$ is the dual map of $\mathcal{E}_F$, we define a code $\hat{\mathcal{C}}=(\{\mathcal{E}^m\},\{\mathcal{E}_F^*(\Gamma^m)\})$ in the encoder-decoder pair $\in(\mathfrak{E}_p,\mathfrak{D}_{\mathrm{sep}})$ for state $\Psi_{AF}^{\otimes n}$. Since $s(\mathcal{C})=s(\hat{\mathcal{C}})$, Eq. (73) follows from Eq. (D44).

Next, we show Eq. (74). For a state ρ'_{AF}, we have a trace-preserving operation $\mathcal{E}'_F\in\mathscr{C}(F\to F)_{\mathrm{ppt}}$ such that $\rho_{AF}=\mathcal{E}'_F(\Psi_{AF})$. Let $\mathcal{C}=(\{\mathcal{E}^m\},\{\Gamma^m\})\in(\mathfrak{E}_{\mathrm{ppt}},\mathfrak{D}_{\mathrm{ppt}})$ be a code for state ${\rho'_{AF}}^{\otimes n}$. Since $\{{\mathcal{E}'_F}^*(\Gamma^m)\}$ is a separable measurement, we define a code $\hat{\mathcal{C}}'=(\{\mathcal{E}^m\},\{{\mathcal{E}'_F}^*(\Gamma^m)\})$ in the encoder-decoder pair $\in(\mathfrak{E}_{\mathrm{ppt}},\mathfrak{D}_{\mathrm{ppt}})$ for state $\Psi_{AF}^{\otimes n}$. Since $s(\mathcal{C})=s(\hat{\mathcal{C}}')$, Eq. (74) follows from Eq. (D43). ■

Proof. First, we show Eq. (75). Let $\mathcal{C}=(\{\mathcal{E}^m\},\{\Lambda^x\},\{\Gamma^{\widehat{m}|x}\})$ be a code in the encoder-decoder pair $(\mathfrak{E}_g,\mathfrak{D}_{\rightarrow})$ for state $\Psi_{AF}^{\otimes n}$. Since the operation $\mathcal{E}_F$ is a trace-preserving positive operation, the operation ${\mathcal{E}_F}^*$ is a unit-preserving positive operation. Since $\{{\mathcal{E}_F}^*(\Lambda^x)\}$ is a POVM on $\mathcal{H}_F^{\otimes n}$, we define a code $\hat{\mathcal{C}} = (\{\mathcal{E}^m\}, \{{\mathcal{E}_F}^*(\Lambda^x)\}, \{\Gamma^{\widehat{m}|x}\})$ in the encoder-decoder pair $(\mathfrak{E}_g, \mathfrak{D}_\rightarrow)$ for state $\rho_{AF}^{\otimes n}$. Since $s(\mathcal{C}) = s(\hat{\mathcal{C}})$, Eq. (75) follows from Eq. (D38).

Next, we show Eq. (76). Let $\mathcal{C} = (\{\mathcal{E}^{m'}\}, \{\Lambda^{x'}\}, \{\Gamma^{\widehat{m}|x'}\})$ be a code in the encoder-decoder pair $(\mathfrak{E}_g, \mathfrak{D}_\rightarrow)$ for state ${\Psi'_{AF}}^{\otimes n}$. Since $\{{\mathcal{E}'_F}^*(\Lambda^x)\}$ is a POVM on $\mathcal{H}_F^{\otimes n}$, we define a code $\hat{\mathcal{C}}' = (\{\mathcal{E}^{m'}\}, \{{\mathcal{E}'_F}^*(\Lambda^{x'})\}, \{\Gamma^{\widehat{m}|x'}\})$ in the encoder-decoder pair $(\mathfrak{E}_g, \mathfrak{D}_\rightarrow)$ for state ${\rho'_{AF}}^{\otimes n}$. Since $s(\mathcal{C}) = s(\hat{\mathcal{C}}')$, Eq. (76) follows from Eq. (D38). ■

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