Quantum Advantage in Information Retrieval

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Random access codes have provided many examples of quantum advantage in communication, but concern only one kind of information retrieval task. We introduce a related task—the Torpedo Game—and show that it admits greater quantum advantage than the comparable random access code. Perfect quantum strategies involving prepare-and-measure protocols with experimentally accessible three-level systems emerge via analysis in terms of the discrete Wigner function. The example is leveraged to an operational advantage in a pacifist version of the strategy game *Battleship*. We pinpoint a characteristic of quantum systems that enables quantum advantage in any bounded-memory information retrieval task. While preparation contextuality has previously been linked to advantages in random access coding we focus here on a different characteristic called sequential contextuality. It is shown not only to be necessary and sufficient for quantum advantage, but also to quantify the degree of advantage. Our perfect qutrit strategy for the Torpedo Game entails the strongest type of inconsistency with noncontextual hidden variables, revealing logical paradoxes with respect to those assumptions.

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I. INTRODUCTION

Random access coding involves the encoding of a random input string into a shorter message string. The encoding should be such that any element of the input string can be retrieved with high probability from the message string. Such tasks have long been studied as examples in which the communication of quantum information can provide advantage, i.e., enhanced performance, over classical information, e.g., Refs. [1–8].

However, random access coding concerns only one kind of information retrieval. In this work we introduce another such task—the Torpedo Game. It is similar to random access coding, but with additional requirements involving the retrieval of relative information about elements of the input string. Taking a geometric perspective it may also be viewed as a pacifist version of the popular strategy game *Battleship* (see Fig. 1). Quantum strategies can be implemented in prepare-and-measure scenarios, and outperform classical strategies for the Torpedo Games with bit and trit inputs. In particular, quantum perfect strategies exist in the trit case and provide a greater quantum advantage than for the comparable random access coding task [5].

Optimal quantum strategies emerge from an analysis in terms of the discrete Wigner function. Wigner negativity is a signature of nonclassicality in quantum systems that is related to contextuality and that has been widely studied as a resource for quantum speedup and advantage [9–16]. Knowing which characteristic lies at the source of better-than-classical performances can both allow for comparison of quantum systems in terms of their utility, and offer a heuristic for generating further examples of quantum-enhanced performance. Our optimal quantum strategies are indeed Wigner negative, with perfect quantum strategies derived from maximum Wigner negativity. Yet while negativity is necessary for advantage in the Torpedo game, it is not sufficient.

To more precisely pinpoint the source of quantum advantage we must look further. One candidate would be preparation contextuality [17], another signature of nonclassicality that has been linked to quantum advantages (e.g., Ref. [18]) and has also been linked to quantum random access codes (QRACs) in numerous studies [3,6,19]. It has been shown to be necessary for advantage in a

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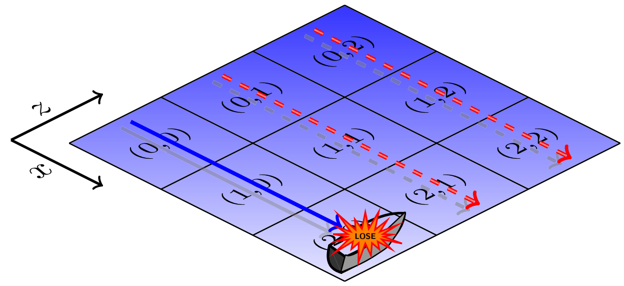

FIG. 1. The Torpedo Game is a pacificist alternative to *Battleship* where the aim is to avoid sinking Alice's ship, depicted here in dimension 3.

restricted class of random access codes subject to an obliviousness constraint [16,20].

In this work, however, we focus on a different characteristic called sequential contextuality [21]. It indicates the absence of a hidden variable model respecting the sequential structure of a given protocol. Subject to an assumption of bounded memory, we find that this characteristic is necessary and sufficient for quantum advantage, not just in random access coding but in any information retrieval task. Moreover, we show that it quantifies the degree of advantage that can be achieved.

Contextuality can exhibit itself at the level of probability distributions, e.g., quantum violations of the Clauser-Horne-Shimony-Holt (CHSH) inequality [22]. But in some cases it can also manifest at the level of the supports of these distributions. In other words contextuality can be inferred by a series of logical deductions about which events are possible or not, e.g., Hardy's paradox [23,24]. This situation has come to be known as logical contextuality [25]. In the most extreme cases, known as strong contextuality [25], every possible event triggers such a paradox [26], e.g., Popescu-Rohrlich box violations of the CHSH inequality [27] (see also Refs. [28,29]). Note that empirical data is always expressible as a convex mixture of classical and strongly contextual data [25,30]. For qutrits the quantum perfect strategies we introduce display analogous contextuality, and hence paradoxes, of this strongest form in a prepare-and measure scenario.

Section II gives an overview of information retrieval tasks including random access coding and the Torpedo Game. Section III provides background on discrete Wigner functions. Section IV deals with optimal classical and quantum strategies for the Torpedo Game; a presentation on how this game can be used for a dimensional witness; and a derivation of the nonlocal version of the Torpedo Game. Finally, Sec. V establishes the relationship between sequential contextuality and quantum advantage in bounded-memory information retrieval tasks.

II. INFORMATION RETRIEVAL TASKS

We begin with a recap of the familiar example of random access codes before generalizing to the broader notion of information retrieval tasks.

A. Random access codes

An $(n,m)_2$ random access code (RAC), sometimes denoted $n \to m$, is a communication task in which one aims to encode information about a random n-bit input string into an m-bit message, where $m < n$, in such a way that any one of the input bits may be retrieved from the message with high probability. An $(n,m)_2$ QRAC instead encodes the input into an m-qubit (quantum) message state.

Such tasks may be considered as two-party cooperative games in which the first party, Alice, receives a random input string from a referee. She encodes information about this in a message that is communicated to the second party, Bob. The referee then asks Bob to retrieve the value of the bit at a randomly chosen position in the input string. We assume that the referee's choices are made uniformly at random.

For instance, for the $(2,1)_2$ RAC game [1] an optimal classical strategy is for Alice to directly communicate one of the input bits to Bob. If asked for this bit, Bob can always return the correct answer, otherwise Bob guesses and will provide the correct answer with probability $\frac{1}{2}$. Thus the game has a classical value of

$$\theta^C_{2\to 1} = \frac{1}{2}\left(1+\frac{1}{2}\right) = \frac{3}{4}.$$

Quantum strategies can outperform this classical bound.

An optimal quantum strategy is for Alice to communicate the qubit state

$$\left|\psi_{x_1,x_2}\right\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + \frac{1}{\sqrt{2}}[(-1)^{x_1} + (-1)^{x_2} i]\,|1\rangle\right), \quad (1)$$

where (x,z) is the input bit string she has received. Bob measures in the X basis when asked for x_1 and in the Y basis when asked for x_2 (see Fig. 2). If he obtains the $+1$ eigenvalue he returns the value 1 and if he obtains the -1 eigenvalue he returns 0. This yields a quantum value for the game of

$$\theta^Q_{2\to 1} = \cos^2\left(\frac{\pi}{8}\right) \approx 0.85.$$

B. General information retrieval tasks

One may also consider more general communication scenarios. In an $(n,m)_d$ communication scenario the input is a random string of n $d$its and the message is a string of m (qu)$d$its, for $d \geq 2$. (Q)RAC tasks have previously been considered in such scenarios, e.g., in Refs. [5,31].

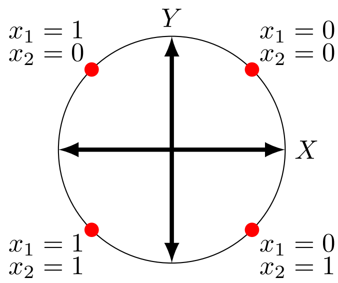

FIG. 2. The four red dots correspond to the four states $|\psi_{x_1 x_2}\rangle$ defined in Eq. (1) depicted as points on the equator of the Bloch sphere.

However, we also wish to accommodate for a much wider range of information retrieval tasks. An information retrieval task in an $(n, m)_d$ communication scenario is specified by a tuple $\langle Q, \{w_q\}_{q\in Q}\rangle$.

(a) Q is a finite set of *questions*.
(b) The $w_q : \mathbb{Z}_d^n \to \mathbb{Z}_d$ are winning relations, which pick out the good answers to question q given an input string in $\mathbb{Z}_d^n$. Note that there may be more than one good answer, or none. It is assumed that inputs and outputs are endowed with the structure of the commutative ring $\mathbb{Z}_d$.

Standard $(n, m)_d$ (Q)RACs are recovered when the questions ask precisely for the respective input dits. In that case the winning relations w_i reduces simply to projectors onto the respective dits of the input string. However, other interesting tasks arise when the questions also concern relative information about the input string, in the form of parities or linear combinations modulo d of the input dits. Similar generalizations for $d = 2$, using functions rather than relations, have been independently proposed in Refs. [32,33].

C. The Torpedo Game

Of particular interest in the present work is an information retrieval task for $(2, 1)_d$ communication scenarios. We take the game perspective and refer to the task as the dimension d Torpedo Game (see Fig. 1 in dimension 3 and Fig. 3). Let x and z be the two input dits. There are $d + 1$ questions $Q = \{\infty, 0, 1, \ldots, d - 1\}$. The labeling comes from a geometric interpretation to be elaborated upon shortly. Winning relations for the Torpedo Game are given by

$$
\begin{aligned}
w_\infty(x, z) &= \{a \in \mathbb{Z}_d \mid a \neq x\},\\
w_0(x, z) &= \{a \in \mathbb{Z}_d \mid a \neq -z\},\\
w_1(x, z) &= \{a \in \mathbb{Z}_d \mid a \neq x - z\}, \qquad (2)\\
w_2(x, z) &= \{a \in \mathbb{Z}_d \mid a \neq 2x - z\},\\
&\vdots\\
w_{d-1}(x, z) &= \{a \in \mathbb{Z}_d \mid a \neq (d - 1)x - z\}.
\end{aligned}
$$

All arithmetic is modulo d.

For $d = 2$, the Torpedo Game is equivalent to a $(2, 1)_2$ (Q)RAC, but with an additional question. Bob may be asked to retrieve either one the individual input dits, or to retrieve relative information about them in the form of their parity $x \oplus z$. We note that dimension 2 is the only case where the winning relations are actually functions (there is only one good answer per question).

The Torpedo Game may be framed as a cooperative, pacifist alternative to the popular game *Battleship*, in which Alice and Bob, finding themselves on opposing sides in a context of naval warfare, wish to subvert the conflict and cooperate to avoid casualities while not directly disobeying orders.

We take the input dits received by Alice as designating the coordinates in which she is ordered by her commander to position her one-cell ship on the affine plane of order d. We may think of the affine plane as a toric $d \times d$ grid, with x designating the row and z the column. For example, in Fig. 4 we identify the top edge with the bottom edge and the left edge with the right edge.

Bob is a naval officer on the opposing side who is ordered by his commander to shoot a torpedo along a line

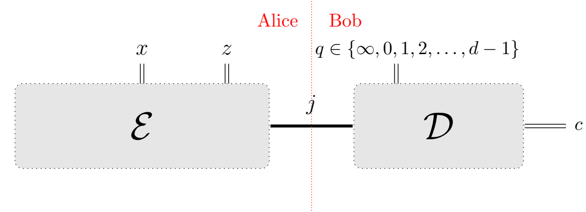

FIG. 3. Prepare-and-measure protocol for the Torpedo Game: Alice receives dits x and z and sends a single message (qu)dit j via the encoding $\mathcal{E}$. Bob is asked a question $q \in \{\infty, 0, \ldots, d - 1\}$, performs decoding $\mathcal{D}$, and outputs c, which should satisfy the winning conditions given by $w_q(x, z)$ with high probability.

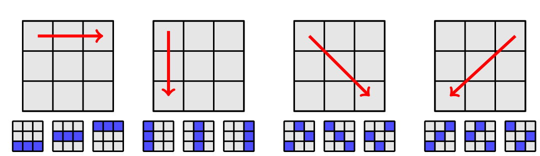

FIG. 4. The red arrows depict the directions or slopes ($\infty, 0, 1, 2$, respectively) along which Bob may be asked to shoot in the $d = 3$ Torpedo Game. For each direction, Bob has three possibilities, depicted by the blue lines. In the affine plane of order 3, each of these groups of three blue cells forms a line.

of the grid with slope specified by $q \in Q$. The ∞ question requires Bob to shoot along some row, and the 0 question requires Bob to shoot along some column, etc. However, Bob retains the freedom to choose which row, or column, or diagonal of given slope, as the case may be. In other terms, upon receiving q Bob must shoot along lines $qx - z = c$ (if $q \neq \infty$) or $x = c$ (if $q = \infty$) but is free to choose the constant c.

Alice and Bob wish to coordinate a strategy for avoiding casualities, while still obeying their explicit orders. Tight security measures on Alice's side mean that she can only transmit a single (qu)dit of information. Based on the (qu)dit received, Bob must choose his c in such a way that he avoids Alice's ship.

III. THE DISCRETE WIGNER FUNCTION

It is possible to represent finite-dimensional quantum states as quasiprobability distributions over a phase space of discrete points. Wootters [34,35] introduced a method of constructing discrete Wigner functions (DWFs) based on finite fields, wherein vectors from a complete set of mutually unbiased bases in $\mathbb{C}^d$ are put in one-to-one correspondence with the lines of a finite affine plane of order d. This geometric picture of the DWF is useful for visualizing our Torpedo Game as exemplified in Fig. 4, where each distinct orthonormal basis corresponds to a set of d parallel (nonintersecting) lines. Gross [36] singled out one particularly symmetric definition of the DWF that obeys the discrete version of Hudson's theorem. This theorem says that an odd-dimensional pure state is non-negatively represented in the DWF if and only if it is a stabilizer state (defined below). The discrete Hudson's theorem has remarkable implications, providing large classes of quantum circuit with a local hidden variable model that enables efficient simulation [10,37]. Clearly, negativity in this DWF is a necessary prerequisite for quantum speedup. Howard *et al.* [11] showed that this negativity actually corresponds to contextuality with respect to Pauli measurements, thereby establishing the operational utility of contextuality for the gate-based model of quantum computation (particularly in a fault-tolerant setting). The equivalence of Wigner negativity and contextuality was established by deriving a noncontextuality inequality using the graph-theoretic techniques of Cabello *et al.* [38], which extend Kochen-Specker-type state-independent proofs to the state-dependent realm. This proof (and a subsequent alternate proof [39]) requires that, as well as the system displaying Wigner negativity, a second ancillary system must be present in order to have a sufficiently rich set of available measurements. Very recent work by Schmid *et al.* [40] reinforces the special nature of the Gross DWF by identifying it as the unique Spekkens noncontextual ontological model for the stabilizer subtheory of quantum mechanics, thereby proving the necessity of Spekkens' contextuality for quantum computation (the necessity of Kochen-Specker contextuality having already been established in Ref. [11]).

A. DWF formalism

The discrete Wigner function is both foundationally interesting as well as practically relevant for fault-tolerant quantum computing via its link with so-called "stabilizer states." The qudit versions of the X and Z Pauli operators are

$$X|k\rangle = |k+1\rangle,$$
$$Z|k\rangle = \omega^k|k\rangle,$$

where $\omega = \exp(2\pi i/d)$ and arithmetic is modulo d. The qudit Pauli group has elements that are products of (powers of) these operators, e.g., X^xZ^z for $x, z \in \mathbb{Z}_d$. A unitary U stabilizes a state $|\psi\rangle$ if $U|\psi\rangle = |\psi\rangle$. A stabilizer state is the unique n-qudit state stabilized by a subgroup of size d^n of the Pauli group. Equivalently, stabilizer states may be understood as the image of computational basis states under the Clifford group, which is the set of unitaries that map the Pauli group to itself under conjugation.

For an arbitrary $d \times d$ Hermitian operator Q of unit trace (typically a density matrix), its Wigner representation will consist of d^2 real quasiprobabilities $W_{x,z}$ for $x, z \in \mathbb{Z}_d$. In particular, the quasiprobability associated with the point $(x, z) \in \mathbb{Z}_d^2$ is given by

$$W_{x,z} = \frac{1}{d}\mathrm{Tr}(QA_{x,z}),$$

where $A_{x,z}$ are the so-called phase-point operators to be defined shortly. The unit trace of Q will ensure that $\sum_{x,z} W_{x,z} = 1$. Taking the magnitude $|W_{x,z}|$ of each quasiprobability will lead to $\sum_{x,z} |W_{x,z}| = 1$ if and only if the quasiprobability distribution is actually a legitimate (non-negative) discrete probability distribution. In contrast, the presence of negative quasiprobabilities entails $\sum_{x,z} |W_{x,z}| > 1$, and in fact the departure of $\sum_{x,z} |W_{x,z}|$ from unity is a sensible measure of "how negative" or "how nonclassical" the DWF of an operator is [10,41].

When working with the DWF, it is convenient to use the Weyl-Heisenberg notation and phase convention for the qudit Pauli operators, i.e.,

$$D_{x,z} = \omega^{2^{-1}xz} \sum_k \omega^{kz}|k+x\rangle\langle k| = \omega^{xz/2}X^xZ^z,$$

where they go by name displacement operators. The phase-point operator at the origin of phase space $A_{0,0}$ is given by the simple expression

$$A_{0,0} = \sum_{j\in\mathbb{Z}_d} |-j\rangle\langle j|,$$

and the remainder are found by conjugation with displacement operators

$$A_{x,z} = D_{x,z} A_{0,0} D_{x,z}^{\dagger}. \tag{3}$$

B. DWF and information retrieval

The eigenvectors of phase-point operators are objects of interest. The maximizing eigenvectors of the phase-point operators in Eq. (3) (and additional ones from different choices of DWF) were used in Casaccino *et al.* [31] as the encoded messages of a $(d+1,1)_d$ QRAC. This is natural given the use of mutually unbiased bases (MUBs) in constructing DWFs, and prominence of MUBs in the QRAC literature. If Alice receives input $\mathbf{k} = (k_1, k_2, \dots, k_{d+1}) \in \mathbb{Z}_d^{d+1}$ that she encodes in $\rho_{\mathbf{k}}$ and transmits to Bob, then the average probability of success for the Casaccino *et al.* QRAC is

$$\frac{1}{(d+1)d^{d+1}} \sum_{\mathbf{k}\in\mathbb{Z}_d^{d+1}} \mathrm{Tr}\left[\rho_{\mathbf{k}}(\Pi_1^{k_1} + \cdots + \Pi_{d+1}^{k_{d+1}})\right], \tag{4}$$

where Π_q^i is the projector corresponding to dit value i in Bob's qth measurement setting. Since phase-point operators are constructed using sums of projectors from MUBs, i.e., $\Pi_1^{k_1} + \Pi_2^{k_2} + \cdots + \Pi_{d+1}^{k_{d+1}}$, the use of a maximizing eigenvector of a phase-point operator for $\rho_{\mathbf{k}}$ is natural to maximize Eq. (4).

In this work we instead make use of the *minimizing* eigenvectors of phase-point operators. The rationale for this is twofold: (i) these eigenvectors display remarkable geometric properties with respect to the measurements in (their constituent) mutually unbiased bases, and (ii) negativity (of a state in the DWF) is the hallmark of nonclassicality, which has already been identified with contextuality (with the already mentioned caveat that an additional "spectator" subsystem was required). These will be seen to lead to a perfect quantum strategy for the Torpedo Game.

As previously noted in Refs. [36,42], the eigenvectors of phase point operators Eq. (3) are degenerate: a $+1$ eigenspace of dimension $(d+1)/2$ and a -1 eigenspace of dimension $(d-1)/2$. Any state in the -1 eigenspace has an outcome that is forbidden [42,43] in each of a complete set of MUBs. For example, let $|\psi_{0,0}\rangle = (|1\rangle - |d-1\rangle)/\sqrt{2}$ satisfying $A_{0,0}|\psi_{0,0}\rangle = -|\psi_{0,0}\rangle$. This state obeys $\mathrm{Tr}(\Pi_q^0 |\psi_{0,0}\rangle\langle\psi_{0,0}|) = 0$, where Π_q^0 is the projector on the 0th eigenvector in the qth basis. More specifically, Π_q^0 is the projector corresponding to the $\omega^0 = +1$ eigenvector of displacement operator $\{D_{0,1}, D_{1,0}, D_{1,1}, \dots, D_{1,d-1}\}$. These displacement operators have eigenvectors leading to mutually unbiased measurement bases $q \in \{\infty, 0, 1, \dots, d-1\}$, respectively. The related states $|\psi_{x,z}\rangle = D_{x,z}|\psi_{0,0}\rangle$, which are eigenstates $A_{x,z}|\psi_{x,z}\rangle = -|\psi_{x,z}\rangle$, obey

$$\mathrm{Tr}\left[|\psi_{x,z}\rangle\langle\psi_{x,z}| \left(\Pi_\infty^x + \Pi_0^{-z} + \Pi_1^{x-z} + \cdots + \Pi_{d-1}^{(d-1)x-z}\right)\right] = 0. \tag{5}$$

Equation (5) implies that the probability of the relevant outcome (outcome x in the first basis, $-z$ in the second basis, etc.) in each of the MUBs is zero: cf. Eq. (2). The general expression for odd power-of-prime d is proven in Refs. [44,45].

IV. OPTIMAL STRATEGIES FOR THE TORPEDO GAME

Here we gather the optimal classical, quantum, and (in one case) postquantum strategies for the Torpedo Game. We focus only on Torpedo Games with power-of-prime dimension d as we are able to provide perfect quantum strategies in these cases (for $d \geq 3$) due to the fact that there exist $d+1$ MUBs for those dimensions. The quantum case differs depending on whether we use a qubit or a qudit of odd prime-power dimension. The classical optimum can only be established rigorously for small dimensions, owing to the proliferation of possible hidden variable assignments as the dimension increases. We obtain a quantum advantage for dimension 2 and 3. At the conclusion of this paper we sketch a modified Torpedo Game that we believe may have a lower classical value whenever $d \geq 5$, thereby re-establishing a quantum advantage in those dimensions.

A. Optimal quantum and postquantum strategies

1. Quantum perfect strategy for odd power-of-prime dimension

From Eq. (5) it follows that there is a perfect quantum strategy for the dimension d Torpedo Game for any odd power-of-prime d:

(1) Upon receiving dits x and z Alice sends the following state to Bob:

$$|\psi_{x,z}\rangle = D_{x,z}|\psi_{0,0}\rangle = D_{x,z}\left[\sqrt{2}^{-1}(|1\rangle - |-1\rangle)\right]. \tag{6}$$

(2) Bob receives $|\psi_{x,z}\rangle$ and is asked a question $q \in \{\infty, 0, \dots, d-1\}$. He measures the state in the MUB corresponding to q and outputs the dit corresponding to the measurement outcome.

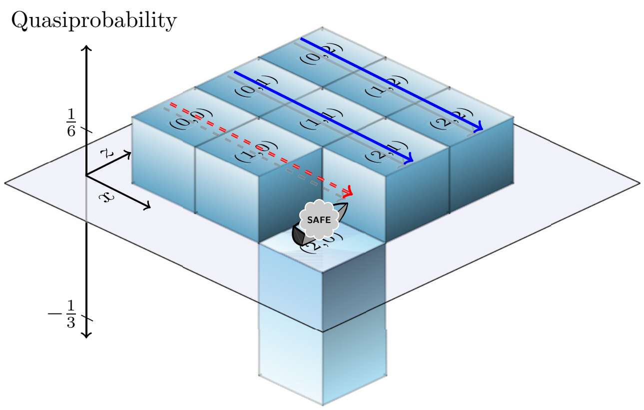

FIG. 5. The perfect quantum strategy can be understood by plotting the discrete Wigner function of the message state sent by Alice. In the above qutrit case, Alice is in coordinate $(x,z) = (2,0)$, so she sends Bob the state $|\psi_{2,0}\rangle$ whose Wigner function is $-1/3$ at coordinate $(2,0)$ and $1/6$ otherwise. Bob measures this state along any of the four allowed directions, wherein the probability of each outcome is given by sum of quasiprobabilities along the corresponding line. Hence, the only outcomes with nonzero probability of occurring correspond to lines not passing through $(2,0)$. Whichever outcome Bob sees, he may fire his torpedo along the corresponding line, safe in the knowledge that it will not intersect Alice's ship. In this figure the solid blue lines correspond to the possible outcomes for the $q = 0$ direction, but the same argument holds for all the other directions.

This quantum strategy wins the Torpedo Game deterministically, i.e., with probability 1. In Fig. 5 we provide geometric intuition for why this strategy is perfect.

2. Optimal quantum strategy for qubits

An analogous strategy to the qudit case can be employed for the qubit Torpedo Game, using message states $|\psi_{x,z}\rangle = X^x Z^z |\psi_{0,0}\rangle$ where X, Y, and Z are the usual qubit Pauli spin matrices and $|\psi_{0,0}\rangle\langle\psi_{0,0}| = \frac{1}{2}\left[\mathbb{I} - (X+Y+Z)/\sqrt{3}\right]$.
For $d = 2$, while this does not constitute a perfect strategy it still achieves an advantage over classical strategies. In fact, it turns out to be an optimal strategy: this strategy achieves a winning probability of approximately 0.79 and we show that this is optimal. First we can leverage the fact that the $(3,1)_2$ (Q)RAC attributed to Isaac Chuang is at least as hard to win as the Torpedo Game because for the last question, instead of asking to retrieve the parity of the input bits, the $(3,1)_2$ (Q)RAC ask to retrieve a third independent bit. Thus we get a lower bound of $\frac{1}{2}\left(1 + 1/\sqrt{3}\right) \approx 0.79$ on the optimal quantum value. To obtain a matching upper bound, we implement numerically the Navascués-Pironio-Acín (NPA) hierarchy [46], which is a hierarchy of semidefinite programs converging from the exterior to the correlations arising from quantum systems. Because the message sent from Alice to Bob is of finite dimension we rely mostly on Ref. [47], which allows characterization of correlations arising from finite-dimensional quantum systems. We find a matching upper bound proving that indeed $\theta^{Q}_{d=2} \approx 0.79$.

3. Perfect postquantum strategy for qubits

The average probability of success for the Casaccino *et al.* QRAC, see Eq. (4), can be maximized by using a postquantum "state" of the form $\Pi_1^{k_1} + \Pi_2^{k_2} + \cdots + \Pi_{d+1}^{k_{d+1}} - \mathbb{I}$, where scare quotes reflect the fact that, although it is Hermitian and has unit trace, its spectrum is not necessarily non-negative. In fact the "state" above is a phase-point operator, $A_{\mathbf{k}}$, for one of Wootters' discrete Wigner functions. Since phase-point operators obey $\mathrm{Tr}(A\mathbb{I}) = 1$ and $\mathrm{Tr}(AA) = d$ then $1/(d+1)d^{d+1}\sum_{\mathbf{k}\in\mathbb{Z}_d^{d+1}} \mathrm{Tr}\left[A_{\mathbf{k}}(A_{\mathbf{k}} + \mathbb{I})\right] = 1$. In other words, in the same spirit as Ref. [48], there is a perfect strategy by using postquantum states. Seen in this way, phase-point operators in a $(d+1,1)_d$ QRAC scenario are similar to Popescu-Rohrlich [27] boxes in the CHSH scenario. As seen above, our Torpedo Game has a perfect strategy within quantum mechanics for all odd power-of-prime dimensions, by construction. In contrast, we saw that the qubit Torpedo Game only has quantum value of roughly 0.79. To reach a perfect strategy, we must once again use a phase-point operator as the nonphysical "state" that Alice sends to Bob, see Fig. 6.

B. Optimal classical strategies

In what follows, we describe an encoding map $\mathcal{E} = \{p_{\mathcal{E}}(\cdot|x,z)\}_{x,z}$ as specifying a probability distribution over

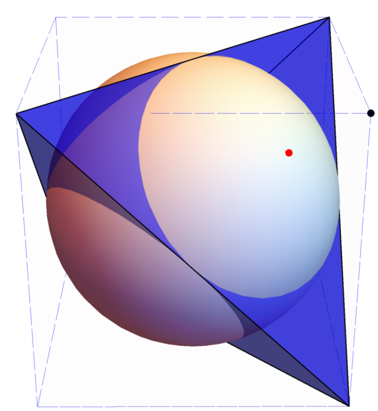

FIG. 6. The qubit version of the Torpedo Game has a perfect strategy when allowed access to postquantum "states." The red point on the surface of the Bloch sphere represents the optimal message state $|\psi_{0,0}\rangle$, achieving a value of 0.79 for the Torpedo Game. The black point representing $\frac{1}{2}\left[\mathbb{I} - (X+Y+Z)\right]$ is not a valid density matrix, but achieves a value of 1 in the Torpedo Game.

messages $j \in \mathbb{Z}_d$ for each combination of inputs $x, z \in \mathbb{Z}_d$. Similarly a decoding map $\mathcal{D} = \{p_{\mathcal{D}}(\cdot|j, q)\}_{j,q}$ specifies a probability distribution over outputs $c \in \mathbb{Z}_d$ for each combination of a message and question, $j, q \in \mathbb{Z}_d$, respectively.

Combining an encoding $\mathcal{E}$ and a decoding $\mathcal{D}$ results in an empirical behavior $e = \{p_e(\cdot|x, z, q)\}_{x,z,q}$. This is a set of probability distributions over outputs $c \in \mathbb{Z}_d$, one for each combination of the referee variables $x, z \in \mathbb{Z}_d$, $q \in Q$ such that

$$p_e(c|x, z, q) = \sum_{j \in \mathbb{Z}_d} p_{\mathcal{D}}(c|j, q)\, p_{\mathcal{E}}(j\,|x, z). \tag{7}$$

By comparison, quantum-mechanical empirical behaviors arise via the Born rule: $p_e(c|x, z, q) = \mathrm{Tr}(\rho_{x,z}\Pi_q^c)$.

Assuming the referee variables to be uniformly distributed, a strategy has a winning probability given in terms of its empirical probabilities as

$$\frac{1}{d^2(d+1)} \sum_{x,z,q} p_e[w_q(x, z) \mid x, z, q].$$

The classical value of the Torpedo Game for dimension d can thus be expressed as

$$\theta_d^C = \max_{\mathcal{E},\mathcal{D}} \left[\frac{1}{d^2(d+1)} \sum_{x,z,q} p_e[w_q(x, z) \mid x, z, q] \right]. \tag{8}$$

For evaluation of this expression note that it suffices to consider deterministic encodings and decodings. In the presence of shared randomness, nondeterministic strategies can always be obtained as convex combinations of deterministic ones and the expression is convex linear [49]. Furthermore, for each encoding there exists a decoding that is optimal with respect to it. This fact was also observed for one-way communication tasks with messages of bounded dimension in Ref. [16]. Thus it is possible to evaluate the classical value by enumerating over deterministic encodings only.

Proposition 1. *The classical value of an information retrieval task can be expressed as a maximum over encodings as*

$$\theta^C = \max_{\mathcal{E}} \left[\frac{1}{d^2(d+1)} \sum_{j,q} \max_c \sum_{\substack{(x,z)\ \text{s.t.}\\ c \in w_q(x,z)}} p_{\mathcal{E}}(j\,|x, z) \right]. \tag{9}$$

Proof. Starting from Eq. (8),

$$\begin{aligned}
\theta^C &= \max_{\mathcal{E},\mathcal{D}} \left[\frac{1}{d^2(d+1)} \sum_{x,z,q} p_e(w_q(x, z) \mid x, z, q) \right] \\
&= \max_{\mathcal{E},\mathcal{D}} \left[\frac{1}{d^2(d+1)} \sum_{x,z,q} \sum_{c \in w_q(x,z)} p_e(c \mid x, z, q) \right] \\
&= \max_{\mathcal{E},\mathcal{D}} \left[\frac{1}{d^2(d+1)} \sum_{q,c} \sum_{\substack{(x,z)\ \text{s.t.}\\ c \in w_q(x,z)}} p_e(c \mid x, z, q) \right] \\
&= \max_{\mathcal{E},\mathcal{D}} \left[\frac{1}{d^2(d+1)} \sum_{j,q,c} \sum_{\substack{(x,z)\ \text{s.t.}\\ c \in w_q(x,z)}} p_{\mathcal{D}}(c|j, q)\, p_{\mathcal{E}}(j \mid x, z) \right] \\
&= \max_{\mathcal{E}} \left[\frac{1}{d^2(d+1)} \sum_{j,q} \max_c \sum_{\substack{(x,z)\ \text{s.t.}\\ c \in w_q(x,z)}} p_{\mathcal{E}}(j \mid x, z) \right],
\end{aligned}$$

where the last line follows by using a deterministic decoding that is optimal with respect to the encoding. ∎

A useful way of representing any deterministic encoding is as a coloring of the $d \times d$ affine plane using no more than d colors. Observe that a deterministic encoding can alternatively be expressed as a function $f_{\mathcal{E}} : \mathbb{Z}_d \times \mathbb{Z}_d \to \mathbb{Z}_d$, where $f_{\mathcal{E}}(x, z)$ is the message dit to be sent (with probability 1) given inputs x, z. Thinking of the inputs as coordinates in the $d \times d$ affine plane a deterministic encoding is equivalent to a partition of the plane into no more than d equivalence classes, or a coloring using no more than d colors.

1. Optimal strategies for d = 2 and d = 3

In general there are d^{d^2} partitions of a $d \times d$ grid. For low dimensions the expression in Eq. (9) can be evaluated by exhaustive search over partitions. For dimension 2 and 3 we find

$$\theta_{d=2}^C = \frac{3}{4} \quad \text{and} \quad \theta_{d=3}^C = \frac{11}{12}. \tag{10}$$

Example of strategies that attains these values are depicted below in Fig. 7 and in Fig. 8.

2. Optimal strategies for d = 4 and beyond

As d increases it quickly becomes infeasible to perform an exhaustive search over all partitions. We have, however, found perfect classical strategies, i.e., strategies that win with probability 1, for $d = 4$ (see Fig. 9 for an example in

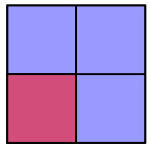

FIG. 7. An optimal classical strategy for the $d = 2$ Torpedo Game. Alice uses her bit of communication to indicate in which class of the partition that she finds herself. Classes are represented here by colors.

dimension 5) up to $d = 23$. This leads us to conjecture that there exists a perfect classical strategy for all $d > 5$.

$$\textbf{Conjecture:} \quad \theta^C_{d\geq 5} = 1. \tag{11}$$

C. Comparison of quantum and classical game values

Recall the optimal quantum values established in Sec. IV A,

$$\theta^Q_{d=2} \simeq 0.79 \quad \text{and} \quad \theta^Q_{d\geq 3} = 1. \tag{12}$$

Comparing these with the classical bounds from Sec. IV B we obtain the ratios

$$\frac{\theta^Q_{d=2}}{\theta^C_{d=2}} \simeq 1.053 \quad \text{and} \quad \frac{\theta^Q_{d=3}}{\theta^C_{d=3}} \simeq 1.091. \tag{13}$$

By comparison, it was shown in Ref. [5] that the classical and quantum values of the $(4,1)_3$ (Q)RAC are $\frac{16}{27}$ and 0.637, respectively, giving a ratio of $\theta^Q_{d=3}/\theta^C_{d=3} \simeq 1.075$. Accordingly, we note that the $d = 3$ Torpedo Game admits a greater quantum-over-classical advantage than the standard random access coding task whose optimal QRAC also exploits the four mutually unbiased bases available in dimension 3.

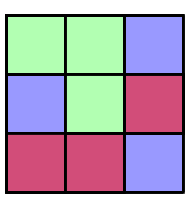

FIG. 8. An optimal classical strategy for the $d = 3$ Torpedo Game. Alice uses her dit of communication to indicate in which equivalence class (represented by the same colored cells) of the large grid partition she finds herself. The smaller grids (cf. Fig. 4) show where Bob chooses to shoot, given a direction and a color. For the first direction, when asked to shoot horizontally in the grid, notice that Bob may avoid Alice with certainty if she is in either of the red or green partitions. Lines that avoid Alice with certainty are depicted in the corresponding color, whereas black lines intersect with Alice's position with probability $\frac{1}{3}$. Overall, this strategy wins the Torpedo Game with probability $\frac{1}{4}(\frac{8}{9} + \frac{8}{9} + 1 + \frac{8}{9}) = \frac{11}{12}$.

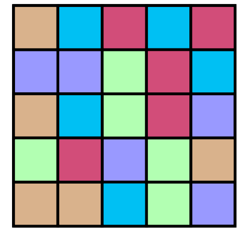

FIG. 9. A perfect classical strategy for the $d = 5$ Torpedo Game. As Fig. 8, the same colored cells belong to the same partition. The lines that avoid Alice are depicted below for every question Bob can be asked.

D. Dimensional witness

The Torpedo Game can be used as a dimension witness for qubits and qutrits. In the following, we modify slightly the setting of the game to allow the message to be of arbitrary dimension. In particular, we no longer require that the message between Alice and Bob is of the same dimension as the inputs. For instance, we allow Alice to send a (qu)trit while she receives two input bits. We thus specify the *dimension of the inputs* as well as the *dimension of the message*. Questions are defined as before and are fixed by the dimension of the inputs: $d + 1$ questions for inputs of dimension d.

Following Table I, we can use the Torpedo Game to discriminate between qubits and qutrits. Moreover these witnesses can distinguish between classical and quantum systems of the same dimension.

E. Nonlocal game

In Ref. [51] a connection was drawn between nonlocal games and single-system games (e.g., prepare-and-measure). That paper concluded by noting the relevance of sequential contextuality and its link to Landauer's principle. In particular, it is very desirable to further understand how nonlocal games can be turned into single-system

TABLE I. Classical and quantum optimal values of the Torpedo Game when we allow the message dimension to differ from that of the inputs. Classical values are computed by exhaustive search. Quantum values are obtained by a combination of a seesaw algorithm and implementation of the Navascués-Vértesi (NV) hierarchy [47] through the interface QDimSum [50]. Note that only the seesaw algorithm succeeded for the trit-input, qubit-message Torpedo Game as the NV hierarchy does not perform well with POVMs.

Input dimension	Message dimension	θ^C	θ^Q
2	2	0.75	0.789
2	3	0.833	0.875
3	2	0.833	>0.867
3	3	0.917	1

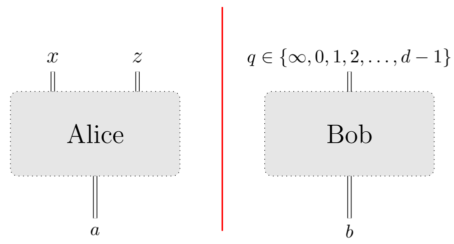

FIG. 10. Schematic of the nonlocal Torpedo Game: Alice and Bob are spacelike separated. Alice receives dits x and z. Bob is asked a question $q \in \{\infty, 0, \ldots, d-1\}$ and outputs c, which should satisfy the winning conditions given by $w_q(x, z)$ with high probability. In the quantum version, they may share a maximally entangled state composed of a d-dimensional quantum state for each party.

games and vice versa. To better analyze this connection, we focus on a nonlocal version of the (qu)trit Torpedo Game that was independently proposed in Refs. [52,53] where nonlocality, steering, and quantum state tomography is carried out in a single experiment. Interestingly, the nonlocal game developed there is exactly the nonlocal version of the dimension 3 Torpedo Game (see Fig. 10). This further provides evidence that this is an interesting route to pursue.

We briefly review Ref. [52] in our language. In this nonlocal version of the dimension-3 Torpedo Game, Alice is no longer allowed to send a (classical or quantum) state to Bob. Because they are spacelike separated, the best classical strategy for Bob is a random strategy. For any inputs $x, z \in \mathbb{Z}_3$ sent to Alice and any question $q \in \{\infty, 0, 1, 2\}$ asked to Bob, he will avoid the wrong answer [and thus satisfy the winning conditions in Eq. (2)] $\frac{2}{3}$ of the time. However, they are allowed to share a maximally entangled state quantumly and they might use this resource to outperform the classical bound. The quantum strategy is the nonlocal version of the optimal strategy for the dimension-3 Torpedo Game.

(a) They share the maximally entangled state

$$\left|\psi^+\right\rangle = \frac{1}{\sqrt{3}} \left(|00\rangle + |11\rangle + |22\rangle\right). \tag{14}$$

(b) Upon receiving trits x and z, Alice measures her part of $\left|\psi^+\right\rangle$ with the phase-point operator $A^T_{x,z}$. With probability $\frac{1}{3}$, she successfully obtains the -1 eigenstate of $A^T_{x,z}$ and consequently steers Bob's state to the -1 eigenstate of $A_{x,z}$ [called $\left|\psi_{x,z}\right\rangle$ in Eq. (5)]. Otherwise, with probability $\frac{2}{3}$ she obtains the $+1$ outcome and steers Bob's state into a $+1$ eigenstate of $A_{x,z}$.

(c) Upon receiving question $q \in \{\infty, 0, 1, 2\}$, Bob measures his steered state in the corresponding qth MUB. If his state was steered into a -1 eigenstate of $A_{x,z}$ then he will always avoid the wrong answer; if his state was steered into a $+1$ eigenstate of $A_{x,z}$, then it will avoid the wrong answer only half of the time. The overall probability of winning the game is $\frac{1}{3}(1) + \frac{2}{3}\left(\frac{1}{2}\right) = \frac{2}{3}$.

The quantum strategy detailed above only wins $\frac{2}{3}$ of the time—the same as the classical strategy—due to the possible projection onto $+1$ eigenstates of $A_{x,z}$ operators. However, if one were to postselect on the -1 outcome of Alice there would be a local description. [(repetition) and we saw that if one takes the failed outcome of Alice into account the observed correlations are not better than a random strategy.] To circumvent this issue, the following witness of observed correlations is introduced in Ref. [52]:

$$\mathcal{B} = \sum_{b,x,z,q} c_{b,x,z,q}\, p(a = -1, b|x, z, q) - 2 \sum_{x,z} p(a = -1|x, z), \tag{15}$$

where $c_{b,x,z,q} = (-1)^{[b \notin w_q(x,z)]}$ and $[\cdot]$ is the Iverson bracket (for a statement S, $[S] = 1$ if S is true and $[S]$=0 otherwise). The second term is a penalty on Alice's marginal, which penalizes her outputting the successful outcome -1. This gives her incentive not to always output the successful outcome in a hidden-variable description. The magnitude of the second term (the coefficient -2) was fine-tuned so that the best-trade-off in a hidden-variable description is for Alice to output -1 only $\frac{1}{3}$ of the time matching exactly the quantum strategy. The crucial inequality obtained in Ref. [52] is the following:

$$\mathcal{B} \overset{\text{LHV}}{\leq} 4 \overset{\text{Quantum}}{\leq} 6. \tag{16}$$

Local strategies can only reach a value of 4 for $\mathcal{B}$ while sharing a maximally entangled state allows a value of 6 to be reached. Note that the quantum strategy detailed above reaches the bound of 6. Remarkably, the Bell inequality so obtained is a facet of the local hidden-variable polytope, or "tight" as it is often called.

V. CHARACTERIZING AND QUANTIFYING CONTEXTUALITY IN INFORMATION RETRIEVAL TASKS

When quantum advantage is observed in a bounded-memory information retrieval task like a (Q)RAC task or the Torpedo Game, it highlights a difference between the information carrying capacities of qudits compared to dits for a fixed dimension. It can be remarked that such a difference is a consequence of the different geometries of the respective state spaces. In this section, however, we seek a sharper, quantified analysis of the source of the advantage in terms of contextuality.

We use the notion of sequential contextuality that was introduced in Ref. [21] to extend structural treatments of Bell-Kochen-Specker contextuality [25] to sequential operational scenarios. As such, it is a behavioral characteristic that can arise in experiments involving sequences of operations. While Ref. [21] was concerned specifically with sequences of transformations, here we take a broader view that also includes the operations of preparation and measurement. In the special case of prepare-and-measure scenarios, sequential contextuality recovers a natural notion of classicality in terms of realizability by hidden variables. For instance, in the bounded-memory regime we are interested in whether sequential contextuality also matches the characteristic introduced by Żukowski in Ref. [54].

We note that sequential contextuality is distinct from the notion of contextuality due to Spekkens [17], as discussed in Ref. [21]. It is also separate from the analyses of Refs. [55–58], which sought to close potential loopholes created by sequentiality of measurements in experimental tests of the more traditional Bell-Kochen-Specker form of contextuality. That said, it may be possible to view the latter analyses as providing a plausible mechanism for sequential contextuality, though it is not a perspective we pursue here.

The study of contextuality arose in quantum foundations, where a major theme is the attempt to understand empirical behaviors that may appear nonintuitive from a classical perspective, e.g., the Einstein-Podolsky-Rosen paradox [59]. The typical approach is to look for a description of physical systems at a deeper level than the quantum one at which more classically intuitive properties may be restored. Such a description is usually formalized as a hidden-variable model for the behavior (sometimes also referred to as an ontological model [17]). The great significance of the celebrated no-go theorems of quantum foundations, like Bell's theorem [60] and the Bell-Kochen-Specker theorem [61,62], was to prove that certain "nonclassical" features of the empirical behaviors of quantum systems are necessarily inherited by any underlying model.

Nonclassical features of quantum systems like contextuality are also increasingly investigated for their practical utility. For instance, in previous work involving the present authors, contextuality of the Bell-Kochen-Specker kind was shown to be a prerequisite for quantum speedup [11] and to quantify quantum-over-classical advantage in a variety of informational tasks [30].

Bell-Kochen-Specker contextuality essentially concerns the statistics that arise under varied measurements on a physical system. In contrast, our notion of contextuality concerns the statistics that arise from sequences of operations—preparations, transformations, and measurements—all of which can vary. As a behavioral feature sequential contextuality signifies the absence of any hidden-variable model that would preserve a compositional description of operations performed in sequence. In other words, sequential *non*contextuality requires a hidden-variable model in which each operation has an independent, modular description (as a transformation on hidden variables) such that to describe a sequence of operations one simply composes their hidden-variable descriptions. We provide a more rigorous mathematical description in the following subsections.

Rather than focusing on characteristics that must be inherited by all hidden-variable models, as is common in foundational works, we also take a practical perspective and shift focus to characteristics that must be inherited by bounded-memory models—a constraint that matches the informational problem at hand. In this respect, the significance of sequential contextuality in what follows can also be viewed through a practical rather than a foundational lens, as a characteristic that quantifies quantum advantage.

A. Empirical behaviors and hidden-variable models

Recall from Sec. IV B that any strategy for an information retrieval task gives rise to an empirical behavior $e = \{p_e(\cdot|i,q)\}_{i,q}$. In other words, for each combination of input string $i \in \mathbb{Z}_d^n$ and question $q \in Q$ there is a resulting probability distribution over outputs. This is true regardless of whether the strategy is classical, quantum, postquantum, or other. The combination of input string and question fully specifies the precise operations that are performed in the sequence. This is what we refer to as the context, just as a context in a (Bell-Kochen-Specker) measurement scenario specifies a set compatible measurements to be performed jointly. We have also chosen a formal description of empirical behavior that echoes the formalism of empirical models for measurement scenarios in Ref. [25].

Given an empirical behavior, one can ask whether it can be simulated by a bounded-memory hidden-variable model. In particular, we are interested in models that respect the sequential structure of the strategy, bearing in mind that the inputs and questions specify a sequence of operations: either preparations, transformations, or measurements. Our main focus is on prepare-and-measure scenarios, in which sequences arise from a combination of a preparation and a measurement. To match the constraints of information retrieval tasks, memory is bounded by the dimension of the message string. This is further motivated by the Holevo bound [63], according to which one can faithfully retrieve no more than n dits of classical information from n qudits.

In a bounded-memory model, the hidden variable is restricted to take values in $\mathbb{Z}_d$ with d fixed by the communication scenario. A state preparation P is modeled by a probability distribution $p_\pi(\cdot \mid P)$ over the hidden-variable space $\mathbb{Z}_d$. Similarly, a measurement M is modeled by

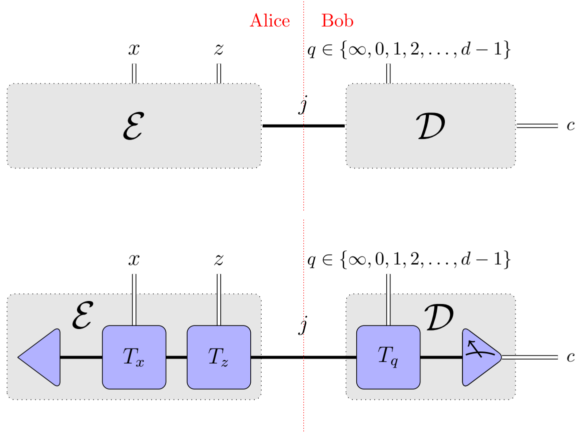

FIG. 11. The Torpedo Game in a prepare-and-measure (top) versus transformational scenario (bottom). The empirical models $e_{x,z,q}$ in either scenario are the same both quantumly and classically. Hence classical and quantum strategies for any $(2,1)_d$ information retrieval task can be equivalently expressed in prepare-and-measure or transformational form.

a family of probability distributions $\{p_\mu(\cdot \mid \lambda, M)\}_\lambda$ over the outcome space, which to match the communication scenario is also $\mathbb{Z}_d$. A hidden-variable model, e.g., for a prepare-and-measure sequence of operations $M_q \circ P_{x,z}$ as in Fig. 11, simulates an empirical behavior as

$$p_e(\cdot|x,z,q) = \sum_\lambda p_\mu(\cdot|\lambda, M_q) \cdot p_\pi(\lambda \mid P_{x,z}). \qquad (17)$$

Each operation is thus modeled in sequence as an operation on the hidden-variable space.

With Eq. (17) in mind, the bounded-memory classical strategies of Sec. IV can also be interpreted as bounded-memory hidden-variable models themselves. To see this, note that in Eq. (7) encoding corresponds to hidden-variable preparation and decoding to hidden-variable measurement.

While the above description makes contact with the strategies and empirical behaviors of Sec. IV, it will be convenient for the remainder of this section to use a simplified notation. For all preparations $P_{x,z}$, the probability distribution $p_\pi(\cdot \mid P_{x,z})$ will be more concisely denoted as a probability vector $\boldsymbol{\lambda}_{x,z}$ in $\mathbb{R}^d$. For measurements, we express $\{p_\mu(\cdot \mid \lambda, M_q)\}_\lambda$ more concisely as a set $\left\{\mathbf{v}_q^c\right\}_{c\in\mathbb{Z}^d}$ of non-negative real vectors, one for each outcome $c \in \mathbb{Z}_d$. To ensure that the total probability distributed over outcomes sums to 1, the vectors must satisfy the property that for all q, $\sum_{c\in\mathbb{Z}_d} \mathbf{v}_q^c = \mathbf{1}$ where $\mathbf{1} = (1,1,\ldots,1)^T$. If we also denote by $e_{x,z,q} := p_e(\cdot \mid x,z,q)$ the empirical probability vector over outcomes $\mathbb{Z}_d$, then Eq. (17) can be rewritten for each outcome $c \in \mathbb{Z}_d$ in simplified notation as the dot product

$$e_{x,z,q}(c) = \mathbf{v}_q^c \cdot \boldsymbol{\lambda}_{x,z}. \qquad (18)$$

Here our focus is on prepare-and-measure scenarios, but we note that transformations can similarly be represented as left-stochastic matrices. For more on transformations and how measurements may be viewed as a kind of transformation see Sec. A 1.

For our main example of an information retrieval task we have also focused on the prepare-and-measure version of the Torpedo Game. Note, however, that it can be equivalently expressed in a sequential scenario with fixed preparation and measurement (see Fig. 11). In Appendix A we provide the explicit quantum and classical strategies for this purely transformational version of the Torpedo Game as well as the equivalence between prepare-and-measure and sequential protocols.

B. Sequential contextuality in information retrieval tasks

An empirical behavior is sequential noncontextual [21] if it admits a hidden-variable model that (i) preserves a modular sequential description of operations, and (ii) the hidden-variable representation of operations is context independent.

These assumptions have been implicitly built into the above definition of hidden-variable models. For (i), note that each operation has an individual description at the hidden-variable level. For example, to obtain predictions for a prepare-and-measure experiment we compose the individual hidden-variable descriptions of the preparation and of the measurement, as in Eq. (18). And for (ii), note for example that regardless of which context the preparation $P_{x,z}$ appears in it should be modeled by the same vector $\boldsymbol{\lambda}_{x,z}$. One could relax these assumptions, in which case it would become trivial to find a hidden-variable model for any behavior, but it would also entail giving up the intuitive sense of what the model means.

If an empirical behavior does not admit a sequential noncontextual hidden-variable model it is said to be sequential contextual. In this paper we consider only sequential contextuality with respect to bounded-memory models, though the definition may be applied more generally. For a description in terms of transformations see Sec. A 1.

A useful intuition for sequential contextuality is that, within the memory constraints, for any faithful model of the behavior the whole (the description of the context) is more than the composition of its parts (the description of the individual operations). A contextual model would always involve additional memory and communication to track the context, which would be outside of the constraints of the task—involving, e.g., a contextuality demon analogous to Maxwell's demon in thermodynamics. Indeed it was shown in Ref. [51] that a related characteristic incurs a simulation cost as measured by Landauer erasure.

C. Quantifying contextuality via the contextual fraction

For any fixed $(n,m)_d$ communication scenario empirical behaviors are closed under contextwise convex combinations—a property that is inherited from probability distributions. In operational terms, if shared randomness is used to choose between several strategies the result is still an empirical behavior.

For any empirical behavior e, we can consider convex decompositions of the form

$$e = \omega e^{\mathrm{NC}} + (1-\omega)e', \tag{19}$$

where e^{NC} and e' are empirical behaviors for the same task and e^{NC} is noncontextual. The maximum of ω over all such decompositions is referred to as the noncontextual fraction of e, written $\mathrm{NCF}(e)$. Similarly, the contextual fraction of e is $\mathrm{CF}(e) := 1 - \mathrm{NCF}(e)$.

This provides a measure of contextuality in the interval $[0,1]$, where $\mathrm{CF}(e)=0$ indicates that e is noncontextual, $\mathrm{CF}(e)>0$ indicates that e is contextual, and $\mathrm{CF}(e)=1$ indicates that e is maximally contextual (also referred to as strong contextuality).

The contextual fraction was used as a measure for sequential contextuality in Ref. [21]. It extends a natural measure for Bell-Kochen-Specker contextuality [25], which itself generalizes measures based on Bell-inequality violations [30].

D. Quantified contextual advantage in information retrieval tasks

The following proposition can be understood as a no-go theorem stating that winning the Torpedo Game deterministically for $d=2$ and $d=3$ is incompatible with the assumptions of sequential noncontexutality and bounded memory. If such a performance is observed then one is forced to abandon at least one of the assumptions, and we note that the Holevo bound gives an argument that perhaps the noncontextuality assumption is the weaker of these.

Proposition 2. *For $d=2$ and $d=3$, strong sequential contextuality with respect to bounded memory is necessary and sufficient to win the Torpedo Game deterministically.*

Proof. Suppose a bounded-memory hidden-variable model realizes an empirical model that wins the Torpedo Game deterministically. Input-question combinations (x,z,q) label the contexts. Recall that the wining relation is $\omega_q(x,z)$, and the winning condition for the Torpedo Game is

$$p_e(c \notin \omega_q(x,z) \mid x,z,q) = 0. \tag{20}$$

Using notations introduced in Sec. V A, the hidden-variable model must specify probability vectors $\{\boldsymbol{\lambda}_{x,z}\}_{x,z\in\mathbb{Z}_d}$ and non-negative vectors $\left\{\mathbf{v}_q^c\right\}_{c\in\mathbb{Z}^d,q\in Q}$ such that $\sum_{c\in\mathbb{Z}_d}\mathbf{v}_q^c = \mathbf{1}$ for each $q\in Q$, and

$$\mathbf{v}_q^c \cdot \boldsymbol{\lambda}_{x,z} = 0, \tag{21}$$

for all $x,z\in\mathbb{Z}_d, \forall q\in Q$ and $c\notin\omega_q(x,z)$.

As mentioned in Sec. IV B, it suffices to consider deterministic strategies. Equation (21) reduces to a set of binary linear equations (36 equations for $d=3$) that any sequentially noncontextual realization must jointly satisfy.

This cannot be possible since it would provide a perfect classical strategy for the $d=2$ and $d=3$ Torpedo Games, violating the optimal bounds Eq. (10) that were obtained by exhaustive search. On the other hand, it is always possible to obtain a contextual realization, by taking contextwise solutions to Eq. (21): for example, where the choice of $\boldsymbol{\lambda}_{x,z}$ is not only a function of x and z, but also of q.

It can further be observed that if any fraction of an empirical model e can be described noncontextually, i.e., $\mathsf{NCF}(e)=p>0$, then with an average probability at least p the empirical model e fails in the Torpedo Game. Therefore, to win the Torpedo Game deterministically requires strong contextuality. ■

An explicit noncontextual memory-bounded hidden-variable model that fails to fully realize the empirical predictions but that satisfies the maximum of 33 out of 36 constraints from Eq. (21) for $d=3$ is the following. Measurement and state vectors are given in terms of the computational basis vectors, where $\mathbf{f}_k$ denotes the kth computational basis vector (zero indexed) in the vector space $\mathbb{Z}_2^d$ over $\mathbb{R}$. The measurement vectors are

$$\mathbf{v}_q^c = \begin{cases} \mathbf{f}_{2c\oplus 2}, & q=\infty \\ \mathbf{f}_{c\oplus 2}, & q=0,1 \\ \mathbf{f}_{2c\oplus 1}, & q=2 \end{cases}.$$

The state vectors are

$$\boldsymbol{\lambda}_{0,0}=\boldsymbol{\lambda}_{0,1}=\boldsymbol{\lambda}_{1,1}=\mathbf{f}_0,$$
$$\boldsymbol{\lambda}_{1,0}=\boldsymbol{\lambda}_{0,2}=\boldsymbol{\lambda}_{2,2}=\mathbf{f}_1,$$
$$\boldsymbol{\lambda}_{2,0}=\boldsymbol{\lambda}_{2,1}=\boldsymbol{\lambda}_{1,2}=\mathbf{f}_2.$$

This corresponds to the strategy depicted in Fig. 8.

We also obtain the following more general result, of which Proposition 2 is a special case.

Theorem 3. *Given any information retrieval task and strategy with empirical behavior e,*

$$\varepsilon \geq NCF(e)\,\nu,$$

where ε is the probability of failure, averaged over inputs and questions, $NCF(e)$ is the bounded-memory noncontextual fraction of e with memory size d fixed by the scenario, and $\nu := 1 - \theta^C$ measures the hardness of the task, θ^C being the classical value.

Proof. The empirical behavior can be decomposed as

$$e = \mathsf{NCF}(e)e^{\mathrm{NC}} + \mathsf{CF}(e)e',$$

where e' is necessarily strongly contextual. From this convex decomposition, we obtain that the probability of success using the empirical model e reads

$$p_{S,e} = \mathsf{NCF}(e)p_{S,e^{\mathrm{NC}}} + \mathsf{CF}(e)p_{S,e'},$$

where $p_{S,e^{\mathrm{NC}}}$ and $p_{S,e'}$ are the average probabilities associated with empirical models e^{NC} and e', respectively. At best, e' wins with probability 1 and thus

$$p_{S,e} \leq \mathsf{NCF}(e)p_{S,e^{\mathrm{NC}}} + \mathsf{CF}(e),$$
$$\varepsilon \geq \mathsf{NCF}(e)\varepsilon_{e^{\mathrm{NC}}},$$

where $\varepsilon_{e^{\mathrm{NC}}} = 1 - p_{S,e^{\mathrm{NC}}}$ is the average probability of failure associated with e^{NC}. Since the latter is noncontextual, we know that the minimum probability of failure is $\nu = 1 - \theta^C$, where θ^C is the classical value of the game. Then $\varepsilon_{e^{\mathrm{NC}}} \geq \nu$, from which we obtain the desired inequality:

$$\varepsilon \geq \mathsf{NCF}(e)\nu.$$

■

This provides a quantifiable relationship between quantum advantage and sequential contextuality. Inequalities of this form are also known to arise for a variety of other informational tasks that admit quantum advantage, with hardness measures and notions of nonclassicality adapted to the particular task [21,30,64].

VI. DISCUSSION

We have formalized a class of information retrieval tasks in communication scenarios, of which the much-studied problem of (quantum) random access coding is a special case. We showed that quantum-over-classical advantage is explained by quantum contextuality. We have identified a distinct information retrieval task that we have presented as the Torpedo Game, which admits a greater quantum-over-classical advantage than the comparable QRAC for qutrits by exploiting Wigner negativity. Remarkably, the qutrit torpedo strategy is maximally contextual, meaning that no fraction of it can be explained by an underlying noncontextual model. By choosing measurements that are associated with a particular discrete Wigner function, Wigner negativity is necessary for a quantum strategy to perform better than a classical one. However, postquantum strategies (e.g., using phase-point operators) might be optimal while remaining Wigner positive. A more thorough investigation of the precise relationship between negativity and advantage will be a topic for future work.

To obtain quantum perfect strategies for the Torpedo Game we have derived a prepare-and-measure scenario for which quantum mechanics exhibits logically paradoxical behavior (with respect to noncontextual hidden-variable assumptions). More generally, we have identified this as a characteristic that quantifies quantum advantage for any bounded-memory information retrieval task.

In the specific case of random access coding, some works have imposed obliviousness constraints as part of the task as opposed to bounded memory. These restrict what information the receiver can be allowed to infer about the input string. Whereas preparation contextuality is known to be necessary and sufficient for quantum advantage in oblivious tasks [16,20], we have shown sequential contextuality to be necessary and sufficient characteristic for bounded-memory tasks.

We briefly comment on possible generalizations of the Torpedo Game. In Eq. (11) we conjectured, based on an explicit proof in $5 \leq d \leq 23$, that there is a perfect classical strategy in all $d \geq 5$ for the Torpedo Game with standard winning conditions as in Eq. (2). In order to reinstate a quantum-over-classical advantage, as we had in dimensions 2 and 3, we may modify the Torpedo Game to make it harder to win classically. Note that the following modifications have no effect on the quantum values, which remain $\theta^Q_{d\geq 3} = 1$. Because the -1 subspace of Gross' phase-point operator A has dimension $(d-1)/2$ it is possible to enlarge Alice's input from d^2 to $d^2(d-1)/2$. Formally, let $0 \leq \ell < (d-1)/2$, so Alice sends Bob $|\psi_{x,z,\ell}\rangle = X^x Z^z [|\ell+1\rangle + |-(\ell+1)\rangle]/\sqrt{2}$, instead of just $|\psi_{x,z,\ell=0}\rangle := |\psi_{x,z}\rangle$ as before. The modification changes a single relation from $w_\infty(x,z) = \{a \in \mathbb{Z}_d \mid a \neq x\}$ to

$$w_\infty(x,z,\ell) = \{a \in \mathbb{Z}_d | a \in (x+\ell+1, x-\ell-1)\},$$

whereas the remaining conditions persist, i.e, $w_q(x,z,\ell) = w_q(x,z)$ in Eq. (2) for $q \in \{0,1,\ldots,d-1\}$. It seems reasonable that such a game, with more restrictive winning conditions, should be harder to win. Indeed, we were unable to find any perfect classical strategy by sampling, although we cannot rule out its existence since we were unable to exhaustively check all classical strategies. More generally, we have motivated how our perfect quantum strategies for this information retrieval task arise from a remarkable geometric feature of maximally negative states [cf. Eq. (5)], and we expect that this insight can be further mined for quantum advantage in future work.

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APPENDIX A: THE TRANSFORMATIONAL VERSION OF THE TORPEDO GAME

1. Transformations in hidden-variable models

The focus in the main text is on prepare-and-measure scenarios, but it can also be useful to have hidden-variable representations of transformations (see Fig. 12). A transformation T is represented by a family of probability distributions $\{p_\tau(\cdot \mid \lambda, T)\}_\lambda$ over the hidden-variable space, one for each "initial" hidden variable $\lambda \in \mathbb{Z}_d$. In simplified vector-space notation a transformation is simply represented by a left-stochastic matrix $\mathcal{T}_T$.

a. Measurements and preparations as transformations

We can also view measurements and preparations as special kinds of transformations. Referring back to Sec. V A, the vectors $\left\{\mathbf{v}_q^c\right\}_{c\in\mathbb{Z}_d}$ describing a measurement M_q

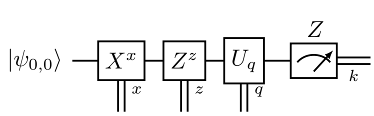

FIG. 12. A perfect strategy in sequential operational form for the dimension d Torpedo Game for odd power-of-prime d. The classically controlled gates are appropriately defined Pauli operators or Clifford gates as in Eq. (A4).

can be gathered into a matrix $\mathcal{T}_q$ with these vectors as rows. By the requirement that $\sum_{c\in\mathbb{Z}_d} \mathbf{v}_q^c = \mathbf{1}$ for all $c \in \mathbb{Z}_d$, the columns of $\mathcal{T}_q$ each sum to 1, as for a left-stochastic matrix, although this measurement matrix is not required to be square. Similarly, the hidden-variable probability vector $\boldsymbol{\lambda}_{x,z}$ corresponding to a preparation (x, z) can be viewed as a matrix whose one column sums to 1. Applying $\mathcal{T}_q$ to a hidden-variable probability vector $\boldsymbol{\lambda}_{x,z}$ then results in the outcome probability vector $e_{x,z,q}$,

$$e_{x,z,q} = \mathcal{T}_q\, \boldsymbol{\lambda}_{x,z},$$

cf. Eq. (18).

b. Sequential noncontextuality in terms of transformations

The structural assumption underlying sequential noncontextuality, that sequential composition of operations must be preserved in the hidden-variable description, is expressible in an appealingly straightforward way in terms of transformations. For example,

$$\mathcal{T}_{T_3\circ T_2\circ T_1} = \mathcal{T}_{T_3}\, \mathcal{T}_{T_2}\, \mathcal{T}_{T_1},$$

or even

$$\mathcal{T}_{M_q\circ T_3\circ T_2\circ T_1\circ P_{x,z}} = \mathcal{T}_q\, \mathcal{T}_{T_3}\, \mathcal{T}_{T_2}\, \mathcal{T}_{T_1}\, \boldsymbol{\lambda}_{x,z},$$

where now the matrix $\mathcal{T}_{M_q\circ T_3\circ T_2\circ T_1\circ P_{x,z}}$ is an outcome probability vector.

2. Equivalence between prepare-and-measure and transformational versions

Any prepare-and-measure strategy for a $(2, 1)_d$ information retrieval task (e.g., the Torpedo Game) can equivalently be re-expressed in a scenario with fixed-state preparation and measurement, and with the classical inputs labeling transformations only. This is depicted in Fig. 11. The equivalence may be useful for experimental implementations, and also makes connections with other transformation-based protocols considered in Refs. [21,51,54,65,66].

Proposition 4. *Classical and quantum strategies for any $(2, 1)_d$ information retrieval task can be equivalently expressed in prepare-and-measure or transformational form.*

Proof. Since the initial preparation is fixed in the sequential version, it is trivial that the encoding step can always be re-expressed as a stochastic map $\mathcal{E} : \mathbb{Z}_d \times \mathbb{Z}_d \to \mathbb{Z}_d$ (or quantumly $\mathcal{E} : \mathbb{Z}_d \times \mathbb{Z}_d \to \mathbb{C}^d$) as in the prepare-and-measure version. Conversely, in the classical case a strategy for the prepare-and-measure version can be expressed as a strategy for the transformational version by setting T_x to always output x and taking for T_z the encoding map $p_{\mathcal{E}}(\cdot|\cdot, z)$ from the classical prepare-and-measure version. In the quantum case T_x outputs $|x\rangle$, and T_z is simply taken to be a Z measurement subsequently composed with the encoding map $\mathcal{E}$. In the hidden-variable description we can fix an arbitrary basis vector, say $\boldsymbol{\delta}_0$, and in the quantum case an arbitrary preparation, say $|0\rangle$.

Similarly, for the decoding step it is trivial that the transformational version can always be expressed as a map in the form of the prepare-and-measure version. For the converse, in the classical case it suffices to take for T_q the stochastic decoding map $p_{\mathcal{D}}(\cdot|\cdot, q)$ from the classical prepare-and-measure version, with fixed measurement given by the identity map (or in the hidden-variable description with measurement simply specified by the basis vectors). In the quantum case, the converse follows from the observation that any projection-valued measurement can be expressed as a unitary transformation followed by a fixed measurement in the Z basis. ■

3. Optimal classical strategy for the transformational version

Note that the following analysis will hold if we consider a global transformation $\mathcal{T}_{x,z}$ instead of two transformations $\mathcal{T}_x$ and $\mathcal{T}_z$ for Alice. A perfect strategy for the Torpedo Game requires that for all $x, z \in \mathbb{Z}_d$, $q \in Q$ and $c \notin \omega_q(x, z)$ that

$$\mathcal{T}_q \mathcal{T}_z \mathcal{T}_x \, \mathbf{f}_0 \cdot \mathbf{f}_c = 0. \tag{A1}$$

For $d = 2$ it was possible to perform a brute-force search over all possible deterministic left-stochastic transformations in order to check how many of the linear equations in Eq. (A1) can be jointly satisfied. As expected, at most 9 out of 12 equations in Eq. (A1) may be jointly satisfied, matching the classical bound of Eq. (10).

For $d = 3$, we were unable to perform the brute-force calculation due to the size of the search space. However, the classical bound of Eq. (10) found by means of our grid partitioning method implies that at most 33 out of 36 equations in Eq. (A1) may be jointly satisfied. A solution that attains the classical value of $\frac{11}{12}$, i.e., that satisfies jointly 33 of the 36 equations from Eq. (A1), using reversible gates only, is the following:

$$\begin{array}{llll} \mathcal{T}_{x=0} = \mathbb{I} & \mathcal{T}_{x=1} = \mathbb{I} & \mathcal{T}_{x=2} = \oplus 1 & \\ \mathcal{T}_{z=0} = \mathbb{I} & \mathcal{T}_{z=1} = \oplus 2 & \mathcal{T}_{z=2} = \oplus 1 & \\ \mathcal{T}_{q=\infty} = \mathbb{I} & \mathcal{T}_{q=0} = \oplus 1 & \mathcal{T}_{q=1} = \oplus 2 & \mathcal{T}_{q=2} = \oplus 1. \end{array} \tag{A2}$$

Reversibility ensures that the strategy does not incur a simulation cost in terms of Landauer erasure, of the kind considered in Ref. [51]. This strategy can also be implemented by states, transformations, and measurements that are non-negatively represented in the discrete Wigner function, taking the stabilizer state $|0\rangle$ as the initial state and representing the above permutation transformations in the obvious way. Thus the classical bound is saturated by a non-negative quantum strategy.

4. Perfect quantum strategy for the Torpedo Game

To explicitly establish a perfect quantum strategy in transformational form for the Torpedo Game, we reestablish the key fact Eq. (5) in the transformational setting. Our proof uses the matrix elements of $A_{x,z}$ combined with the Clifford gates that map the computational basis to each of the additional measurement bases. For this we use the symplectic representation of the Clifford group. The expressions below hold for odd prime d (in the odd prime-power case $d = p^n$ one should replace $\mathbb{Z}_d$ with $\mathbb{F}_d$). Clifford group elements are written as $C = D_{x,z} U_F$ [67] where

$$F = \begin{pmatrix} \alpha & \beta \\ \gamma & \epsilon \end{pmatrix}$$

is an element of the symplectic group $\mathrm{SL}(2, \mathbb{Z}_d)$ (entries of F are in $\mathbb{Z}_d$ and $\det F = 1 \bmod d$), and

$$U_F = \begin{cases} \dfrac{1}{\sqrt{d}} \displaystyle\sum_{j,k=0}^{d-1} \omega^{2^{-1}\beta^{-1}\left(\alpha k^2 - 2jk + \epsilon j^2\right)} |j\rangle\langle k| & \beta \neq 0 \\ \displaystyle\sum_{k=0}^{d-1} \omega^{2^{-1}\alpha\gamma k^2} |\alpha k\rangle\langle k| & \beta = 0. \end{cases}$$

The matrix representation [35] of a phase-point operator is

$$\left(A_{x,z}\right)_{j,k} = \delta_{2x,j+k} \omega^{z(j-k)} \tag{A3}$$

and so $\langle k|A_{x,z}|k\rangle = \delta_{k,x}$ is the likelihood of getting outcome k in a computational basis measurement of $A_{x,z}$. The Clifford unitaries $\{U_\infty, U_0, \ldots, U_{d-1}\}$ that map $Z = D_{0,1}$ to $\left\{D_{0,1}, D_{1,0}, \ldots, D_{1,d-1}\right\}$ are

$$\begin{aligned} \{U_\infty, U_0, \ldots, U_{d-1}\} &= \{\mathbb{I}, HS^0, \ldots, HS^{d-1}\} \\ &= \left\{ U_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)}, U_{\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)}, \ldots, U_{\left(\begin{smallmatrix} d-1 & -1 \\ 1 & 0 \end{smallmatrix}\right)} \right\}, \end{aligned} \tag{A4}$$

where H and S are the qudit versions of the Hadamard and phase gate, respectively. Using Eq. (A3), and the fact that $U_F A_{x,z} U_F^\dagger = A_{x',z'}$ where $\left(\begin{smallmatrix} x' \\ z' \end{smallmatrix}\right) = F \left(\begin{smallmatrix} x \\ z \end{smallmatrix}\right)$ [34,36], it is straightforward to verify that

$$\begin{gathered} \langle k|U_\infty A_{x,z} U_\infty^\dagger|k\rangle = \delta_{k,x} \\ \langle k|U_0 A_{x,z} U_0^\dagger|k\rangle = \delta_{k,-z} \\ \vdots \\ \langle k|U_{d-1} A_{x,z} U_{d-1}^\dagger|k\rangle = \delta_{k,(d-1)x-z}. \end{gathered}$$

For odd prime power $d \geq 3$, the -1 eigenspace of $A_{x,z}$ has rank $(d-1)/2$. We can abuse notation slightly by referring to the normalized projector onto this eigenspace as $|\psi_{x,z}\rangle\langle\psi_{x,z}|$. The final step is to realize that $|\psi_{x,z}\rangle\langle\psi_{x,z}| = 1/(d-1)\left(\mathbb{I} - A_{x,z}\right)$ so that by linearity, and in agreement with Eq. (5) earlier,

$$\begin{aligned}
\mathrm{Tr}\left(|\psi_{x,z}\rangle\langle\psi_{x,z}|\Pi_\infty^k\right) &= \langle k|U_\infty \frac{1}{d-1}\left(\mathbb{I} - A_{x,z}\right)U_\infty^\dagger|k\rangle = \frac{1}{d-1}(1-\delta_{k,x}),\\
\mathrm{Tr}\left(|\psi_{x,z}\rangle\langle\psi_{x,z}|\Pi_0^k\right) &= \langle k|U_0 \frac{1}{d-1}\left(\mathbb{I} - A_{x,z}\right)U_0^\dagger|k\rangle = \frac{1}{d-1}(1-\delta_{k,-z}),\\
&\vdots\\
\mathrm{Tr}\left(|\psi_{x,z}\rangle\langle\psi_{x,z}|\Pi_{d-1}^k\right) &= \langle k|U_{d-1} \frac{1}{d-1}\left(\mathbb{I} - A_{x,z}\right)U_{d-1}^\dagger|k\rangle = \frac{1}{d-1}(1-\delta_{k,(d-1)x-z}).
\end{aligned}$$

Any state in the -1 eigenspace of $A_{x,z}$ wins the Torpedo Game with unit probability, but for concreteness we choose the state Eq. (6).

a. Circuit version of the optimal quantum strategy

As observed in Proposition 4, any quantum strategy for the prepare-and-measure version of the Torpedo Game admits an equivalent strategy for the transformational version. An optimal quantum strategy in sequential operational form takes as fixed preparation $|\psi_{0,0}\rangle$ and as fixed measurement Z. The transformations controlled by x, z, and q are X^x, Z^z, and U_q, respectively, where the unitaries U_q are those defined in Eq. (A4).

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