Model pseudofermionic systems: Connections with exceptional points

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We discuss the role of pseudofermions in the analysis of some two-dimensional models, recently introduced in connection with non-self-adjoint Hamiltonians. Among other aspects, we discuss the appearance of exceptional points in connection with the validity of the extended anticommutation rules which define the pseudofermionic structure.

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I. INTRODUCTION

In recent years, extending what was previously done with canonical commutation relations, one of us (FB) considered a deformed version of the canonical anticommutation relation (CAR) [1], leading to an interesting functional structure: biorthogonal bases $\mathcal{F}_\varphi = \{\varphi_0,\varphi_1\}$ and $\mathcal{F}_\Psi = \{\Psi_0,\Psi_1\}$ appear, as well as lowering, raising, and non-self-adjoint number operators N and $N^\dagger$, whose eigenvectors are exactly the elements in $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$. Also, we find intertwining operators connecting N and $N^\dagger$ which are bounded, invertible, and self-adjoint. The same structure can be extended to more *pseudofermionic modes*, and some applications to optical and electronic systems have also been proposed [2,3].

Here we discuss systematically how the single-mode pseudofermions (PFs) can be naturally used, in the context of some models introduced in recent years in connection with pseudo-Hermitian systems. Among other aspects, we consider exceptional points (EPs), trying to characterize them in terms of our modified CAR. Our main conclusion is that EPs are linked to the absence of PFs: in all of the models considered here, we will show that, in correspondence with their EPs, it becomes impossible to introduce operators satisfying the required anticommutation rules, while, whenever these rules [see (2.1) below] are satisfied, we are away from EPs.

The paper is organized as follows: in the next section, we briefly discuss some basic facts on PFs. Section II A is devoted to a rather general construction, i.e., to the more general non-self-adjoint Hamiltonian, which can be discussed in terms of pseudofermionic operators, whose symmetries are analyzed in Sec. II B. In Sec. III, we show how this general Hamiltonian can be used in some recent examples of 2×2 non-self-adjoint Hamiltonians proposed by Bender, Jones-Smith, Mostafazadeh, and others. Our conclusions are given in Sec. IV.

II. PSEUDOFERMIONS AND HAMILTONIANS

We begin this section by briefly reviewing the main definitions and results concerning single-mode PFs. The extension to higher dimensions will be discussed later on. The starting point is a modification of the CAR $\{c,c^\dagger\} = c\,c^\dagger + c^\dagger\,c = \mathbb{1}$, $\{c,c\} = \{c^\dagger,c^\dagger\} = 0$, between two operators, c and $c^\dagger$, acting on a two-dimensional Hilbert space $\mathcal{H}$. The CAR is replaced here by the following rules:

$$\{a,b\} = \mathbb{1},\ \ \{a,a\} = 0,\ \ \{b,b\} = 0, \tag{2.1}$$

where the interesting situation is when $b \neq a^\dagger$. These rules automatically imply that a nonzero vector φ_0 exists in $\mathcal{H}$ such that $a\,\varphi_0 = 0$, and that a second nonzero vector Ψ_0 also exists in $\mathcal{H}$ such that $b^\dagger\,\Psi_0 = 0$ [1]. In general, $\varphi_0 \neq \Psi_0$.

Let us now introduce the following nonzero vectors:

$$\varphi_1 = b\varphi_0,\ \ \Psi_1 = a^\dagger\Psi_0, \tag{2.2}$$

as well as the non-self-adjoint operators

$$N = ba,\ \ \mathfrak{N} = N^\dagger = a^\dagger b^\dagger. \tag{2.3}$$

We also introduce the self-adjoint operators S_φ and S_Ψ via their action on a generic $f \in \mathcal{H}$:

$$S_\varphi f = \sum_{n=0}^{1}\langle\varphi_n,f\rangle\,\varphi_n,\ \ S_\Psi f = \sum_{n=0}^{1}\langle\Psi_n,f\rangle\,\Psi_n. \tag{2.4}$$

Hence we get the following results, whose proofs are straightforward and will not be given here:

(1)

$$a\varphi_1 = \varphi_0,\quad b^\dagger\Psi_1 = \Psi_0. \tag{2.5}$$

(2)

$$N\varphi_n = n\varphi_n,\quad \mathfrak{N}\Psi_n = n\Psi_n, \tag{2.6}$$

for $n = 0,1$.

(3) If the normalizations of φ_0 and Ψ_0 are chosen in such a way that $\langle\varphi_0,\Psi_0\rangle = 1$, then

$$\langle\varphi_k,\Psi_n\rangle = \delta_{k,n}, \tag{2.7}$$

for $k,n = 0,1$.

(4) S_φ and S_Ψ are bounded, strictly positive, self-adjoint, and invertible. They satisfy

$$\|S_\varphi\| \leqslant \|\varphi_0\|^2 + \|\varphi_1\|^2,\quad \|S_\Psi\| \leqslant \|\Psi_0\|^2 + \|\Psi_1\|^2, \tag{2.8}$$

$$S_\varphi\Psi_n = \varphi_n,\quad S_\Psi\varphi_n = \Psi_n, \tag{2.9}$$

for $n = 0,1$, as well as $S_\varphi = S_\Psi^{-1}$. Moreover, the intertwining relations

$$S_\Psi N = \mathfrak{N} S_\Psi,\quad S_\varphi\mathfrak{N} = N S_\varphi, \tag{2.10}$$

are satisfied.

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The above formulas show that (i) N and $\mathfrak{N}$ behave essentially as fermionic number operators, having eigenvalues 0 and 1, (ii) their related eigenvectors are, respectively, the vectors of $\mathcal{F}_\varphi = \{\varphi_0,\varphi_1\}$ and $\mathcal{F}_\Psi = \{\Psi_0,\Psi_1\}$, (iii) a and $b^\dagger$ are lowering operators for $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$, respectively, (iv) b and $a^\dagger$ are rising operators for $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$, respectively, (v) the two sets $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are biorthonormal, (vi) the *well-behaved* (i.e., self-adjoint, bounded, invertible, with bounded inverse) operators S_φ and S_Ψ map $\mathcal{F}_\varphi$ in $\mathcal{F}_\Psi$, and vice versa, and (vii) S_φ and S_Ψ intertwine between operators which are not self-adjoint.

We refer to [1,2] for further remarks and consequences of these definitions. In particular, for instance, it is shown that $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are automatically Riesz bases for $\mathcal{H}$, and the relations between fermions and PFs are also discussed.

Going back to (2.1), as we have discussed in [1], the only nontrivial possible choices of a and b satisfying these rules are the following:

$$a(1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad b(1) = \begin{pmatrix} \beta & -\beta^2 \\ 1 & -\beta \end{pmatrix},$$

$$a(2) = \begin{pmatrix} \alpha & 1 \\ -\alpha^2 & -\alpha \end{pmatrix}, \quad b(2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

with nonzero α and β, or, maybe more interestingly,

$$a(3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ -\alpha_{11}^2/\alpha_{12} & -\alpha_{11} \end{pmatrix},$$

$$b(3) = \begin{pmatrix} \beta_{11} & \beta_{12} \\ -\beta_{11}^2/\beta_{12} & -\beta_{11} \end{pmatrix},$$

with

$$2\alpha_{11}\beta_{11} - \frac{\alpha_{11}^2\beta_{12}}{\alpha_{12}} - \frac{eta_{11}^2\alpha_{12}}{\beta_{12}} = 1. \tag{2.11}$$

Other possibilities also exist, but they are those in which a and b exchange their roles or those in which a and b are standard fermion operators. Also, these matrices are not really all independent, since $a(1)$ and $b(1)$ can be recovered from $a(3)$ and $b(3)$ taking $\alpha_{11} = 0$, $\alpha_{12} = 1$, $\beta_{11} = \beta$, and $\beta_{12} = -\beta^2$. Notice that this choice satisfies (2.11). Less trivially, we can also recover $a(2)$ and $b(2)$ from $a(3)$ and $b(3)$. In this case, we need to take $\alpha_{11} = \alpha$, $\alpha_{12} = 1$, $\beta_{11} = x$, and $\beta_{12} = -x^2$, and then send x to zero. This means that in order to consider the more general situation, it is enough to use the operators $a(3)$ and $b(3)$, endowed with condition (2.11). From now on, this will be our choice, and we will simply write them a and b.

Remark. For completeness, we have to mention the paper by Bender and Klevansky [4], where similar generalized anticommutation rules were introduced, but with a different perspective.

A. The Hamiltonian

In view of what we have just seen, the most general diagonalizable Hamiltonian which can be written in terms of a and b is obviously the operator

$$H = \omega N + \rho 1\!\!1 = \begin{pmatrix} \omega\gamma\alpha + \rho & \omega\gamma \\ -\omega\gamma\alpha\beta & -\omega\gamma\beta + \rho \end{pmatrix}, \tag{2.12}$$

where ω and ρ, in principle, could be complex numbers, $\alpha = \frac{\alpha_{11}}{\alpha_{12}}$, $\beta = \frac{\beta_{11}}{\beta_{12}}$, and $\gamma = \alpha_{12}\beta_{11} - \alpha_{11}\beta_{12} = \alpha_{12}\beta_{12}(\beta - \alpha)$. Then we can write

$$a = \alpha_{12}\begin{pmatrix} \alpha & 1 \\ -\alpha^2 & -\alpha \end{pmatrix}, \quad b = \beta_{12}\begin{pmatrix} \beta & 1 \\ -\beta^2 & -\beta \end{pmatrix},$$

while condition (2.11) can be written as $-\gamma^2 = \alpha_{12}\beta_{12}$. This also implies that $(\alpha - \beta)\gamma = 1$.

The eigensystem of H is trivially deduced: the eigenvalues are $\epsilon_0 = \rho$ and $\epsilon_1 = \omega + \rho$, which are real if and only if ρ and ω are both real. In this case, ϵ_0 and ϵ_1 are also the eigenvalues of $H^\dagger = \omega N^\dagger + \rho 1\!\!1$. From now on, except when explicitly stated, we will assume that $\epsilon_j \in \mathbb{R}$, for $j = 0,1$. It might be interesting to notice that by adopting the same limiting procedure described above ($\alpha_{11} = \alpha$, $\alpha_{12} = 1$, $\beta_{11} = x$, $\beta_{12} = -x^2$, and $x \to 0$), we simply recover $H = \rho 1\!\!1$.

The eigenvectors of N and $N^\dagger$, and of H and $H^\dagger$ as a consequence, are the following:

$$\varphi_0 = N_\varphi\begin{pmatrix} 1 \\ -\alpha \end{pmatrix}, \quad \varphi_1 = b\varphi_0 = \frac{\gamma N_\varphi}{\alpha_{12}}\begin{pmatrix} 1 \\ -\beta \end{pmatrix}, \tag{2.13}$$

and

$$\Psi_0 = N_\Psi\begin{pmatrix} 1 \\ \overline{\beta}^{-1} \end{pmatrix}, \quad \Psi_1 = a^\dagger\Psi_0 = \frac{\overline{\gamma}\, N_\Psi}{\overline{\beta_{11}}}\begin{pmatrix} \overline{\alpha} \\ 1 \end{pmatrix}, \tag{2.14}$$

where $N_\varphi\overline{N_\Psi} = \frac{\alpha_{12}\beta_{11}}{\gamma}$. This choice is dictated by the fact that $\langle\Psi_0,\varphi_0\rangle = 1$. Let us remind the reader that φ_0 and Ψ_0 are (almost) fixed by requiring that they are annihilated by a and $b^\dagger$, respectively: $a\varphi_0 = 0$ and $b^\dagger\Psi_0 = 0$. Moreover, we have $N\varphi_j = j\varphi_j$ and $N^\dagger\Psi_j = j\Psi_j$, $j = 0,1$, so that

$$H\varphi_j = \epsilon_j\varphi_j, \quad H^\dagger\Psi_j = \epsilon_j\Psi_j, \tag{2.15}$$

$j = 0,1$. Sometimes it can be useful to write H and $H^\dagger$ in terms of the projectors P_j defined as $P_j f = \langle\Psi_j,f\rangle\varphi_j$, $j = 0,1$, whose adjoint is $P_j^\dagger f = \langle\varphi_j,f\rangle\Psi_j$ clearly.[1] Here, f is a generic vector in $\mathcal{H}$. Then, $H = \epsilon_0 P_0 + \epsilon_1 P_1$ and $H^\dagger = \epsilon_0 P_0^\dagger + \epsilon_1 P_1^\dagger$.

It is a straightforward computation to check that $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ produce, together, a resolution of the identity. Indeed, we have $P_0 + P_1 = P_0^\dagger + P_1^\dagger = 1\!\!1$. Hence, as expected, $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are biorthogonal bases for $\mathcal{H}$.

The next step consists of finding the explicit expressions for S_φ and S_Ψ in (2.4). We find

$$S_\varphi = |N_\varphi|^2\begin{pmatrix} 1 + \left|\frac{\gamma}{\alpha_{12}}\right|^2 & -\overline{\alpha} - \overline{\beta}\left|\frac{\gamma}{\alpha_{12}}\right|^2 \\ -\alpha - \beta\left|\frac{\gamma}{\alpha_{12}}\right|^2 & |\alpha|^2 + \left|\frac{\gamma\beta}{\alpha_{12}}\right|^2 \end{pmatrix} \tag{2.16}$$

and

$$S_\Psi = |N_\Psi|^2\begin{pmatrix} 1 + \left|\frac{\alpha\gamma}{\beta_{11}}\right|^2 & \frac{1}{\overline{\beta}} + \overline{\alpha}\left|\frac{\gamma}{\beta_{11}}\right|^2 \\ \frac{1}{\beta} + \alpha\left|\frac{\gamma}{\beta_{11}}\right|^2 & \left|\frac{1}{\beta}\right|^2 + \left|\frac{\gamma}{\beta_{11}}\right|^2 \end{pmatrix}, \tag{2.17}$$

[1] Of course, they are not orthogonal projectors, since they are not self-adjoint, in general, and not even idempotent.

which are both clearly self-adjoint.[2] Using, for instance, the Sylvester's criterion, it is possible to check explicitly that if $\alpha \neq \beta$, then both S_φ and S_Ψ are positive definite. This can also be deduced looking at the eigenvalues of the two matrices, or just using the definition: $\langle f, S_\varphi f\rangle$ and $\langle f, S_\Psi f\rangle$ are both strictly positive for any nonzero $f \in \mathcal{H}$ if $\alpha \neq \beta$. Interestingly enough, $\alpha = \beta$ implies that condition (2.11) cannot be satisfied, and this means, in turn, that we are losing the pseudofermionic structure described before. In fact, a and b can no longer satisfy the anticommutation rules in (2.1). Therefore, it is not surprising that S_φ and S_Ψ do not admit inverse, contrary to what happens whenever (2.1) are satisfied. We will get a similar conclusion in explicit models: whenever α and β coincide, our operators cannot satisfy (2.11) or its equivalent expressions, and PFs do not appear.

Because of their positivity, there exist unique square-root matrices $S_\varphi^{1/2}$ and $S_\Psi^{1/2}$, which are also positive and self-adjoint. They have a rather involved expression, which we give here for completeness, but which is rather hard to manage:

$$S_\varphi^{1/2} = \frac{|N_\phi|}{\sqrt{2p_1}} \begin{pmatrix} \frac{\sqrt{p_3}\,p_5 - \sqrt{p_2}\,p_4}{2} & \overline{p} \\ p & \frac{\sqrt{p_2}\,p_5 - \sqrt{p_3}\,p_4}{2} \end{pmatrix} \tag{2.18}$$

and

$$S_\Psi^{1/2} = \frac{1}{|N_\phi|\sqrt{2q_1}} \begin{pmatrix} \frac{\sqrt{q_2}\,q_5 - \sqrt{q_3}\,q_4}{2} & \overline{q} \\ q & \frac{\sqrt{q_3}\,q_5 - \sqrt{q_2}\,q_4}{2} \end{pmatrix}, \tag{2.19}$$

where we have defined the following quantities:

$$\begin{aligned}
p_1 &= (1 + t - |\alpha|^2 - t|\beta|^2)^2 + 4|\alpha + t\beta|^2,\\
p_2 &= 1 - \sqrt{p_1} + t + |\alpha|^2 + t|\beta|^2,\\
p_3 &= 1 + \sqrt{p_1} + t + |\alpha|^2 + t|\beta|^2,\\
p_4 &= 1 - \sqrt{p_1} + t - |\alpha|^2 - t|\beta|^2,\\
p_5 &= 1 + \sqrt{p_1} + t - |\alpha|^2 - t|\beta|^2,\\
p &= (\sqrt{p_2} - \sqrt{p_3})(\alpha + t\beta),\\
q_1 &= (|\beta|^2 + |\alpha_{11}|^2 - 1 - |\alpha_{12}|^2)^2 + 4|\beta + \alpha|\alpha_{12}||^2,\\
q_2 &= 1 - \sqrt{q_1} + |\alpha_{11}|^2 + |\alpha_{12}|^2 + |\beta|^2,\\
q_3 &= 1 + \sqrt{q_1} + |\alpha_{11}|^2 + |\alpha_{12}|^2 + |\beta|^2,\\
q_4 &= 1 - \sqrt{q_1} + |\alpha_{11}|^2 - |\alpha_{12}|^2 - |\beta|^2,\\
q_5 &= 1 + \sqrt{q_1} + |\alpha_{11}|^2 - |\alpha_{12}|^2 - |\beta|^2,\\
q &= (\sqrt{q_3} - \sqrt{q_2})(\beta + \alpha|\alpha_{12}|^2),
\end{aligned}$$

and where $t = |\frac{\gamma}{\alpha_{12}}|^2$. Other results which can be explicitly derived are the following:

(1) $S_\varphi \Psi_n = \varphi_n$ and $S_\Psi \varphi_n = \Psi_n$, $n = 0,1$;

(2) $S_\Psi N = N^\dagger S_\Psi$ and $S_\varphi N^\dagger = N S_\varphi$;

(3) calling $c = S_\Psi^{1/2} a\, S_\Psi^{-1/2}$, we find that $c^\dagger = S_\Psi^{1/2} b\, S_\Psi^{-1/2}$ and that $\{c, c^\dagger\} = \mathbb{1}$, $c^2 = 0$;

(4) calling $N_0 = c^\dagger c$, we have $N_0 = S_\Psi^{1/2} N\, S_\Psi^{-1/2} = S_\Psi^{-1/2} N^\dagger\, S_\Psi^{1/2}$;

(5) $e_0 = S_\Psi^{1/2}\varphi_0$ and $e_1 = S_\Psi^{1/2}\varphi_1$ are eigenstates of N_0, with eigenvalues 0 and 1. Therefore, they are also eigenstates of the self-adjoint Hamiltonian $h = S_\Psi^{1/2} H\, S_\varphi^{1/2} = \omega N_0 + \rho\,\mathbb{1}$, with eigenvalues ϵ_0 and ϵ_1. The set $\{e_0, e_1\}$ is an orthonormal basis for $\mathcal{H}$.

All of these results are consequences of the pseudofermionic anticommutation rules in (2.1), and have been deduced and analyzed in [1–3].

B. Symmetry of the Hamiltonian

We continue our analysis of H looking for some nontrivial two-by-two matrix X which commutes with H. Of course, not to make the situation trivial, we assume here that $\omega \neq 0$. Otherwise, $H = \rho\,\mathbb{1}$ and $[H, X] = 0$ for each matrix X. This also happens when $\gamma = 0$, i.e., when $\alpha = \beta$ (not necessarily zero). We recall that, in this last case, we lose the rules in (2.1), so that we are no longer dealing with PFs. This is not a big surprise, since also in this case H turns out to be just a multiple of the identity operator, so that each nonzero vector of $\mathcal{H}$ is an eigenstate of H with eigenvalue ρ.

In case ω and γ are both nonzero, $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ commutes with H only if the following is true: $x_{11} = \frac{x_{12}(\gamma^2\alpha_{11} - \alpha_{12}^2\beta_{12}) + x_{22}\gamma^2\alpha_{12}}{\gamma^2}$, $x_{21} = -x_{12}\alpha\beta$, $x_{22} = x_{11} - x_{12}(\alpha + \beta)$, where x_{11} and x_{12} are free parameters.

Moreover, if we also ask that $X^2 = \mathbb{1}$, we should further require that

$$x_{11} = -x_{22} = \frac{\alpha + \beta}{\alpha - \beta}, \quad x_{12} = \frac{2}{\alpha - \beta}, \quad x_{21} = -\frac{2\alpha\beta}{\alpha - \beta}.$$

Of course, with these choices, also $Y = -X$ commutes with H and satisfies $Y^2 = \mathbb{1}$.

The matrix X can be seen essentially as a generalized version of the $\mathcal{PT}$ symmetry, where

$$\mathcal{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \mathcal{T} := \text{complex conjugate}. \tag{2.20}$$

The Hamiltonian H in (2.12) is not generally $\mathcal{PT}$ symmetric, since the condition $[\mathcal{PT}, H] = 0$ is not guaranteed in general. However, H is $\mathcal{PT}$ symmetric under the following conditions:

$$\rho + \alpha\gamma\omega = \overline{\rho - \beta\gamma\omega}, \quad \alpha\beta\gamma\omega = -\overline{\gamma\omega}, \tag{2.21}$$

and, in this case, the Hamiltonian H becomes

$$H = \begin{pmatrix} \omega\gamma\alpha + \rho & \omega\gamma \\ \overline{\omega\gamma} & \overline{\omega\gamma\alpha + \rho} \end{pmatrix}. \tag{2.22}$$

Here it is more convenient to rewrite its eigenvalues ϵ_0 and ϵ_1 as $\epsilon_\pm = \mathrm{Re}(\rho + \alpha\gamma\omega) \pm \sqrt{Q}$, $Q = |\gamma\omega|^2 - [\mathrm{Im}(\rho + \alpha\gamma\omega)]^2$, and the relative eigenvectors φ_0, φ_1 in (2.13) as

$$|\epsilon_+\rangle = \begin{pmatrix} \frac{i\,\mathrm{Im}(\rho+\alpha\gamma\omega)+\sqrt{Q}}{\overline{\gamma\omega}} \\ 1 \end{pmatrix} = \begin{pmatrix} -\beta^{-1} \\ 1 \end{pmatrix},$$

$$|\epsilon_-\rangle = \begin{pmatrix} \frac{i\,\mathrm{Im}(\rho+\alpha\gamma\omega)-\sqrt{Q}}{\overline{\gamma\omega}} \\ 1 \end{pmatrix} = \begin{pmatrix} -\alpha^{-1} \\ 1 \end{pmatrix},$$

with an obvious notation and with an appropriate choice of normalization. The analytic expression for $\epsilon_\pm$ shows that

[2]Notice that, since $\beta_{11} = \beta_{12}\beta$, we could rewrite S_Ψ using β_{12} rather than β_{11}. This could be useful in the following.

the eigenvalues of H can either be real or form a complex conjugate pair according to the sign of Q.

The $\mathcal{PT}$ symmetry is *unbroken* for $|\gamma\omega| > |\mathrm{Im}(\rho + \alpha\gamma\omega)|$ and, in this case,

$$\mathcal{PT}|\epsilon_+\rangle = \lambda_+|\epsilon_+\rangle, \quad \mathcal{PT}|\epsilon_-\rangle = \lambda_-|\epsilon_-\rangle,$$

with $\lambda_\pm = \frac{\overline{\gamma\omega}}{i\mathrm{Im}(\rho+\alpha\gamma\omega)\pm\sqrt{Q}}$. Notice that $|\lambda_\pm| = 1$, and therefore all of the components of the eigenvectors $|\epsilon_\pm\rangle$ have unitary modulus. This implies that $|\alpha| = |\beta| = 1$. We recall that the eigenvalues of H are actually ρ and $\rho + \omega$, and therefore the *unbroken* $\mathcal{PT}$ symmetry is only compatible with the condition that ρ and ω are both real.

For $|\gamma\omega| < |\mathrm{Im}(\rho + \alpha\gamma\omega)|$, the eigenvalues of H become complex conjugates and the symmetry is *broken* because

$$\mathcal{PT}|\epsilon_+\rangle = \tilde{\lambda}_+|\epsilon_-\rangle, \quad \mathcal{PT}|\epsilon_-\rangle = \tilde{\lambda}_-|\epsilon_+\rangle,$$

with $\tilde{\lambda}_\pm = -i\frac{\overline{\gamma\omega}}{\mathrm{Im}(\rho+\alpha\gamma\omega)\mp\sqrt{\tilde{Q}}}$ and $\tilde{Q} = -Q$. In this case, $|\tilde{\lambda}_\pm| = 1$ and, moreover, $\alpha\overline{\beta} = 1$. The presence of a pair of complex conjugate eigenvalues of H implies necessarily that ρ is imaginary and $\omega = -2i\mathrm{Im}(\rho)$. For $|\gamma\omega| = |\mathrm{Im}(\rho + \alpha\gamma\omega)|$, an EP occurs. The eigenvalues coalesce to the real value $\epsilon = \mathrm{Re}(\rho + \alpha\gamma\omega) = \mathrm{Re}(\overline{\rho - \beta\gamma\omega}) = \rho$ and $|\epsilon_+\rangle = |\epsilon_-\rangle$ which, in turn, implies that $\alpha = \beta$ so that $\gamma = 0$ (we do not consider here the trivial case $\omega = 0$): in this case, the conditions (2.11) are not satisfied and no PFs exist. The formation of an EP is therefore related not only to the absence of the imaginary part of ρ but also to the nonexistence of PFs.

Going back to our matrix X above, it does not, as stated, have the structure of a $\mathcal{PT}$ operator, meaning with this that even if $[X,H] = [\mathcal{PT},H] = 0$, X cannot be identified with $\mathcal{PT}$. This is not a major problem since, in the literature (see, for instance, [5,6]), extended versions of $\mathcal{PT}$ symmetry exist, where it is not required that $[\mathcal{P},\mathcal{T}] = 0$ or that $\mathcal{P} = \mathcal{P}^\dagger$. One such extension has the form

$$\tilde{\mathcal{P}} = \begin{pmatrix} 0 & x \\ 1/x & 0 \end{pmatrix}, \tag{2.23}$$

with $x \neq 0$. If we take x real, the $\tilde{\mathcal{P}}\mathcal{T}$-symmetry condition $[\tilde{\mathcal{P}}\mathcal{T},H] = 0$ is satisfied for the following conditions:

$$\rho + \alpha\gamma\omega = \overline{\rho - \beta\gamma\omega}, \quad x^2\alpha\beta\gamma\omega = -\overline{\gamma\omega}, \tag{2.24}$$

which extend those in (2.21). It is possible to generalize our previous results to this situation: in fact, taking into account (2.24), the eigenvalues of H are $\epsilon_{x_\pm} = \mathrm{Re}(\rho + \alpha\gamma\omega) \pm x^{-2}\sqrt{Q_x}$, and the relative eigenvectors are

$$|\epsilon_{x_+}\rangle = \begin{pmatrix} \frac{ix^2\mathrm{Im}(\rho+\alpha\gamma\omega)+\sqrt{Q_x}}{\overline{\gamma\omega}} \\ 1 \end{pmatrix},$$

$$|\epsilon_{x_-}\rangle = \begin{pmatrix} \frac{ix^2\mathrm{Im}(\rho+\alpha\gamma\omega)-\sqrt{Q_x}}{\overline{\gamma\omega}} \\ 1 \end{pmatrix},$$

where $Q_x = x^2|\gamma\omega|^2 - x^4[\mathrm{Im}(\rho + \alpha\gamma\omega)]^2$. For $Q_x > 0$, we are in the domain of the *unbroken* $\tilde{\mathcal{P}}\mathcal{T}$ symmetry, and the condition $|\alpha| = |\beta| = x^{-2}$ holds. The *broken* $\mathcal{PT}$ symmetry occurs for $Q_x < 0$, and in this case $\overline{\alpha}\beta = x^{-2}$ holds. An EP occurs for $Q_x = 0$, i.e., when $|\gamma\omega| = x^2|\mathrm{Im}(\rho + \alpha\gamma\omega)|$, and as in the specific case of the $\mathcal{PT}$ symmetry, the eigenvalues coalesce to $\epsilon_x = \rho$ and $|\epsilon_{x_+}\rangle = |\epsilon_{x_-}\rangle$, which implies that $\alpha = \beta$ with $\gamma = 0$. This condition is again incompatible with the existence of pseudofermions because (2.11) is no longer verified.

III. EXAMPLES FROM THE LITERATURE

In this section, we show how the above general framework can be used in the analysis of several concrete models introduced over the years by several authors. In other words, we will see that many simple systems considered by many authors fit very well into our framework.

A. An example by Das and Greenwood

The first example we want to consider was discussed in [5] and, in a slightly different version, by others. The Hamiltonian is

$$H_{DG} = \begin{pmatrix} re^{i\theta} & se^{i\phi} \\ te^{-i\phi} & re^{-i\theta} \end{pmatrix}, \tag{3.1}$$

where r,s,t,θ, and ϕ are all real quantities. In particular, to make the situation more interesting, we will assume that r,s, and t are nonzero. We will briefly comment on this possibility later on. H_{DG} coincides with our general H in (2.12) with two different choices of the parameters α, β, ρ, and $\mu = \omega\gamma$:

$$\begin{aligned} &\mu = se^{i\phi}, \\ &\alpha_\pm = ie^{-i\phi}\left\{\frac{r\sin(\theta)}{s} \mp \sqrt{\left[\frac{r\sin(\theta)}{s}\right]^2 - \frac{t}{s}}\right\}, \\ &\beta_\pm = ie^{-i\phi}\left\{\frac{r\sin(\theta)}{s} \pm \sqrt{\left[\frac{r\sin(\theta)}{s}\right]^2 - \frac{t}{s}}\right\}, \\ &\rho_\pm = re^{-i\theta} + is\left\{\frac{r\sin(\theta)}{s} \pm \sqrt{\left[\frac{r\sin(\theta)}{s}\right]^2 - \frac{t}{s}}\right\}. \end{aligned} \tag{3.2}$$

Moreover, the related values of $\omega_\pm$ and $\gamma_\pm$ can be deduced by recalling that, in general, $\gamma = \alpha_{12}\beta_{11} - \alpha_{11}\beta_{12} = \alpha_{12}\beta_{12}(\beta - \alpha)$, $-\gamma^2 = \alpha_{12}\beta_{12}$, and that $(\alpha - \beta)\gamma = 1$. Then, we deduce that whenever $[\frac{r\sin(\theta)}{s}]^2 \neq \frac{t}{s}$,

$$\alpha_{12}\beta_{12} = \frac{e^{2i\Phi}}{4\{[\frac{r\sin(\theta)}{s}]^2 - \frac{t}{s}\}}, \tag{3.3}$$

so that, with a particular choice of the square root,

$$\gamma_\pm = \frac{\pm ie^{i\phi}}{2\sqrt{\{[\frac{r\sin(\theta)}{s}]^2 - \frac{t}{s}\}}}, \tag{3.4}$$

and therefore

$$\omega_\pm = \frac{se^{i\phi}}{\gamma_\pm} = \mp 2is\sqrt{\left\{\left[\frac{r\sin(\theta)}{s}\right]^2 - \frac{t}{s}\right\}}. \tag{3.5}$$

These results show that if $[\frac{r\sin(\theta)}{s}]^2 \neq \frac{t}{s}$, we can *always* recover a pseudofermionic structure for H_{DG}, so that all of the results deduced and listed previously hold true for this model. The situation changes drastically when $[\frac{r\sin(\theta)}{s}]^2 = \frac{t}{s}$. In this case, in fact, $\gamma_\pm = 0$ necessarily, so that (2.11) cannot be satisfied: in this case, no PFs can appear. This is intriguingly related to the existence of EPs in the model, since under this condition the two eigenvalues $E_\pm = r\cos(\theta) \pm \sqrt{st - r^2\sin^2(\theta)}$ of H_{DG} coalesce: $E_+ = E_- = r\cos(\theta)$. We also would like to note that since $s \in \mathbb{R}$, $\omega_\pm$ are real only if $[\frac{r\sin(\theta)}{s}]^2 < \frac{t}{s}$ (*unbroken phase*). On the other hand, if $[\frac{r\sin(\theta)}{s}]^2 > \frac{t}{s}$, ω_+ and ω_- are purely imaginary, and one is the adjoint of the other (*broken phase*).

For completeness, we specialize here the relevant quantities deduced previously. In particular, the eigenvectors of N and $N^\dagger$ are given as in (2.13) and (2.14):

$$\varphi_0^{(\pm)} = N_\varphi \begin{pmatrix} 1 \\ -\alpha_\pm \end{pmatrix}, \quad \varphi_1^{(\pm)} = b\varphi_0^{(\pm)} = \frac{\gamma_\pm N_\varphi}{\alpha_{12}} \begin{pmatrix} 1 \\ -\beta_\pm \end{pmatrix}, \tag{3.6}$$

and

$$\Psi_0^{(\pm)} = N_\Psi \begin{pmatrix} 1 \\ \overline{\beta_\pm}^{-1} \end{pmatrix}, \quad \Psi_1^{(\pm)} = a^\dagger \Psi_0^{(\pm)} = \frac{\overline{\gamma_\pm}\, N_\Psi}{\overline{\beta_{11}}} \begin{pmatrix} \overline{\alpha_\pm} \\ 1 \end{pmatrix}. \tag{3.7}$$

The lowering and raising operators are also *doubled*:

$$a_\pm = \alpha_{12} \begin{pmatrix} \alpha_\pm & 1 \\ -\alpha_\pm^2 & -\alpha_\pm \end{pmatrix}, \quad b_\pm = \beta_{12} \begin{pmatrix} \beta_\pm & 1 \\ -\beta_\pm^2 & -\beta_\pm \end{pmatrix}, \tag{3.8}$$

as well as the operators $S_\varphi^{(\pm)}$ and $S_\Psi^{(\pm)}$, which can be deduced by (2.16) and (2.17) specializing the form of the parameters as in (3.2), (3.3), and (3.4) and writing the following values of α_{12} and β_{11} used also to recover the conditions in (3.2):

$$\begin{aligned} \alpha_{12} &= \frac{2\alpha_{11}\mu}{\mp 2is\sqrt{\{[\frac{r\sin(\theta)}{s}]^2 - \frac{t}{s}\}} + 2ir\sin(\theta)}, \\ \beta_{11} &= \frac{st}{4\left[st - r^2\sin^2(\theta)\right]\alpha_{11}}. \end{aligned} \tag{3.9}$$

Therefore,

$$S_\varphi^{(\pm)} = |N_\varphi|^2 \begin{pmatrix} 1 + \frac{1}{4}\left|\frac{s(x_{rr}^\pm)}{\alpha_{11}\mu\sqrt{x_r}}\right|^2 & \frac{-ie^{i\phi}}{16}\left[4\left|\frac{s(x_{rr}^\pm)^2}{\alpha_{11}\mu\sqrt{x_r}}\right|^2 x_{rr}^\mp + 16x_{rr}^\pm\right] \\ \frac{ie^{-i\phi}}{16}\left[4\left|\frac{s(x_{rr}^\pm)^2}{\alpha_{11}\mu\sqrt{x_r}}\right|^2 x_{rr}^\mp + 16x_{rr}^\pm\right] & |x_{rr}^\pm|^2 + \frac{1}{4}\left|\frac{s(x_{rr}^\pm)^2}{\alpha_{11}\mu\sqrt{x_r}}\right|^2 \end{pmatrix} \tag{3.10}$$

and

$$S_\Psi^{(\pm)} = |N_\Psi|^2 \begin{pmatrix} 1 + 4\left|\frac{s\alpha_{11}x_{rr}^\pm\sqrt{x_r}}{t}\right|^2 & -ie^{+i\phi}\left(\frac{1}{x_{rr}^\mp} + 4x_{rr}^\pm\left|\frac{s\alpha_{11}x_{rr}^\pm\sqrt{x_r}}{t}\right|^2\right) \\ ie^{-i\phi}\left(\frac{1}{x_{rr}^\mp} + 4x_{rr}^\pm\left|\frac{s\alpha_{11}x_{rr}^\pm\sqrt{x_r}}{t}\right|^2\right) & \frac{1}{|x_{rr}^\mp|^2} + 4\left|\frac{s\alpha_{11}x_{rr}^\pm\sqrt{x_r}}{t}\right|^2 \end{pmatrix}, \tag{3.11}$$

where we have introduced $x_r = [\frac{r\sin(\theta)}{s}]^2 - \frac{t}{s}$ and $x_{rr}^\pm = \frac{r\sin(\theta)}{s} \mp \sqrt{x_r}$.

Needless to say, $S_\varphi^{(\pm)}$ and $S_\Psi^{(\pm)}$ have all of the properties we have discussed in Sec. II A and, in particular, they admit square roots $S_\varphi^{(\pm)^{1/2}}$ and $S_\Psi^{(\pm)^{1/2}}$ as in (2.18) and (2.19). For the sake of concreteness, we consider the following particular choice of the parameters of H_{DG}:

$$r = 1,\ s = 0.5,\ t = 1,\ \theta = \phi = \pi/6,$$

and we restrict here to the â–â choice, fixing also $\alpha_{11} = 1$. Then, our operators look like

$$S_\varphi^{(-)} = |N_\varphi|^2 \begin{pmatrix} \frac{1}{2} & -0.317 + 1.549i \\ -0.317 - 1.549i & 3 \end{pmatrix},$$

$$S_\Psi^{(-)} = \frac{1}{2|N_\varphi|^2} \begin{pmatrix} 3 & 0.317 - 1.549i \\ 0.317 + 1.549i & \frac{1}{2} \end{pmatrix}$$

and

$$S_\varphi^{(-)^{1/2}} = |N_\varphi| \begin{pmatrix} 1.076 & -0.117 + 0.572i \\ -0.117 - 0.572I & 1.63 \end{pmatrix},$$

$$S_\Psi^{(-)^{1/2}} = \frac{\sqrt{2}}{2|N_\varphi|} \begin{pmatrix} 1.63 & 0.117 + 0.572i \\ 0.117 + 0.572i & 1.076 \end{pmatrix}$$

and we get

$$\begin{aligned} h_{DG} &= S_\Psi^{(-)^{1/2}} H_{DG}\, S_\varphi^{(-)^{1/2}} \\ &= \begin{pmatrix} 0.832 & 0.393 + 0.306\,i \\ 0.393 - 0.306\,i & 0.9 \end{pmatrix}, \end{aligned}$$

which is the *self-adjoint counterpart* of the Hamiltonian $H_{DG} = \begin{pmatrix} \frac{1}{2}(\sqrt{3}+i) & \frac{1}{4}(\sqrt{3}+i) \\ \frac{1}{2}(\sqrt{3}-i) & \frac{1}{2}(\sqrt{3}-i) \end{pmatrix}$.

Remark. Of course, we can obtain the self-adjoint Hamiltonian h_{DG} only because ρ and ω are real. For the particular values of the parameters in H_{DG} considered here, we obtain $\rho = 1.366$ and $\omega = -1$.

A particular choice of parameters. It is interesting to recall that taking $\phi = 0$ and $s = t$ in H_{DG}, we recover the Hamiltonian

$$H_{part} = \begin{pmatrix} re^{i\theta} & s \\ s & re^{-i\theta} \end{pmatrix},$$

considered, for instance, in [7]. Our previous formulas specialize here in an obvious way. In this case, in particular, EPs are recovered for $\frac{r\sin(\theta)}{s} = \pm 1$. Also,

$$\omega_\pm = \mp 2is\sqrt{\left[\frac{r\sin(\theta)}{s}\right]^2 - 1}$$

is real only if $[\frac{r\sin(\theta)}{s}]^2<1$. Otherwise, ω_+ and ω_- are purely imaginary, and one is the adjoint of the other. EPs appear when $\frac{r\sin(\theta)}{s}=\pm 1$ and, in this case, PFs are absent.

B. A Hamiltonian by Gilary *et al.*

This Hamiltonian was introduced quite recently in [8], and can be rewritten as

$$H_{GMM}=\begin{pmatrix}\epsilon_1-i\Gamma_1 & \nu_0\\ \nu_0 & \epsilon_2-i\Gamma_2\end{pmatrix}, \tag{3.12}$$

where Γ_1 and Γ_2 are positive quantities, ϵ_1 and ϵ_2 are real, and ν_0 is complex valued. It is a simple exercise to show that H_{GMM} can be written as in (2.12) with the following identification:

$$\begin{aligned}
&\omega\gamma=\nu_0,\\
&\alpha_\pm=\frac{1}{2\nu_0}\Big[-\Delta\epsilon+i\Delta\Gamma\mp\sqrt{(-\Delta\epsilon+i\Delta\Gamma)^2+4\nu_0^2}\Big],\\
&\beta_\pm=\frac{1}{2\nu_0}\Big[-\Delta\epsilon+i\Delta\Gamma\pm\sqrt{(-\Delta\epsilon+i\Delta\Gamma)^2+4\nu_0^2}\Big],\\
&\rho_\pm=\frac{1}{2}\Big[\epsilon-i\Gamma\pm\sqrt{(-\Delta\epsilon+i\Delta\Gamma)^2+4\nu_0^2}\Big],
\end{aligned} \tag{3.13}$$

where $\Delta\epsilon=\epsilon_2-\epsilon_1$, $\Delta\Gamma=\Gamma_2-\Gamma_1$, $\epsilon=\epsilon_2+\epsilon_1$, and $\Gamma=\Gamma_2+\Gamma_1$. Since $\gamma_\pm=\alpha_{12}\beta_{12}(\beta_\pm-\alpha_\pm)$ and $\gamma_\pm^2=-\alpha_{12}\beta_{12}$, we find that whenever $\alpha_\pm\neq\beta_\pm$, taking

$$\alpha_{12}\beta_{12}=\frac{-\nu_0^2}{(-\Delta\epsilon+i\Delta\Gamma)^2+4\nu_0^2},$$

the pseudofermionic main condition is satisfied: H_{GMM} admits a pseudofermionic representation. On the other hand, this is not possible if $\alpha_\pm=\beta_\pm$, which is true when $(-\Delta\epsilon+i\Delta\Gamma)^2=-4\nu_0^2$. Looking at the eigenvalues of H_{GMM}, this is exactly the condition which makes its two eigenvalues coalesce. In this case, we have $E_1=E_2=\frac{1}{2}(\epsilon-i\Gamma)$.

The explicit expressions for the relevant eigenvectors and operators can be deduced, as usual, from (2.13), (2.14), (2.16), (2.17), and (3.8).

C. An example by Mostafazadeh and Özcelik

The model we consider now is different from those above because of the absence of EPs. Then, as we will see, it will always be possible to have PFs for all possible values of the parameters of the model.

The Hamiltonian considered in [9] is

$$H_{MO}=E\begin{pmatrix}\cos\theta & e^{-i\phi}\sin(\theta)\\ e^{i\phi}\sin(\theta) & -\cos\theta\end{pmatrix}, \tag{3.14}$$

where $\theta,\phi\in\mathbb{C}$, $\mathrm{Re}(\theta)\in[0,\pi)$, and $\mathrm{Re}(\phi)\in[0,\pi)$. For obvious reasons, we restrict to $E\neq 0$ and to $\theta\neq 0$. We can deduce two different set of values of α, β, etc. for H in (2.12) such that the two Hamiltonians coincide. These choices are

$$\begin{aligned}
&\mu=E\,\sin(\theta)\,e^{i\phi},\quad \alpha_\pm=\frac{e^{i\phi}}{\sin(\theta)}\,[\cos(\theta)\mp 1],\\
&\beta_\pm=\frac{e^{i\phi}}{\sin(\theta)}\,[\cos(\theta)\pm 1],\quad \rho_\pm=\pm E.
\end{aligned} \tag{3.15}$$

The corresponding pseudofermionic operators look like those in (3.8), with

$$\alpha_{12}\beta_{12}=-\tfrac{1}{4}\,\sin^2(\theta)\,e^{-2i\phi}.$$

Also, there exists no possible condition which makes $\gamma_\pm=\alpha_{12}\beta_{12}(\beta_\pm-\alpha_\pm)=0$: contrary to what happens for H_{DG} and for H_{GMM}, this model always allows a pseudofermionic description. The eigenvectors of N and $N^\dagger$ are

$$\begin{aligned}
\varphi_0^{(\pm)}&=N_\varphi\begin{pmatrix}1\\ \frac{e^{i\phi}}{\sin(\theta)}\,[\cos(\theta)\mp 1]\end{pmatrix}\\
\varphi_1^{(\pm)}&=\mp N_\varphi\frac{\cos(\theta)-1}{2\alpha_{11}}\begin{pmatrix}-1\\ \frac{e^{i\phi}}{\sin(\theta)}\,[\cos(\theta)\pm 1]\end{pmatrix},\\
\Psi_0^{(\pm)}&=N_\Psi\begin{pmatrix}1\\ \overline{\left\{\frac{e^{i\phi}}{\sin(\theta)}[\cos(\theta)\pm 1]\right\}^{-1}}\end{pmatrix}\\
\Psi_1^{(\pm)}&=\mp 2N_\Psi\overline{\frac{\alpha_{11}e^{i\phi}}{\sin(\theta)}}\begin{pmatrix}\overline{\frac{e^{i\phi}}{\sin(\theta)}\,[\cos(\theta)\mp 1]}\\ 1\end{pmatrix}.
\end{aligned}$$

In particular, restricting here to the "−" choice, we find that $H_{MO}\varphi_0^{(-)}=-E\varphi_0^{(-)}$, which means that $\epsilon_0^{(-)}=-E$. Moreover, since $\epsilon_1^{(-)}=\epsilon_0^{(-)}+\omega_-$, and since $\omega_-=\frac{\mu}{\gamma_-}=2E$, we deduce that $\epsilon_1^{(-)}=E$. Notice that we have used here

$$\gamma_-=\alpha_{12}\beta_{12}(\beta_--\alpha_-)=\tfrac{1}{2}\,\sin(\theta),e^{-i\phi}.$$

The intertwining operators $S_\varphi^{(-)}$ and $S_\Psi^{(-)}$ now look as follows:

$$S_\varphi^{(-)}=|N_\varphi|^2\begin{pmatrix}1+\left|\frac{\cos(\theta)-1}{2\alpha_{11}}\right|^2 & e^{-i\overline{\phi}}\left(\overline{\frac{[1-\cos(\theta)]\left|\frac{\cos(\theta)-1}{\alpha_{11}}\right|^2-\cos(\theta)-1}{4\sin(\theta)}}\right)\\ e^{i\phi}\left\{\frac{[1-\cos(\theta)]\left|\frac{\cos(\theta)-1}{\alpha_{11}}\right|^2-\cos(\theta)-1}{4\sin(\theta)}\right\} & e^{-2\Im(\phi)}\left\{\left|\frac{[\cos(\theta)-1]^2}{2\alpha_{11}\sin(\theta)}\right|^2+\left|\frac{\cos(\theta)+1}{\sin(\theta)}\right|^2\right\}\end{pmatrix},$$

$$S_\Psi^{(-)}=|N_\Psi|^2\begin{pmatrix}\left|\frac{\alpha_{11}}{\sin^2\left(\frac{\theta}{2}\right)}\right|^2+1 & e^{-i\phi}\overline{\left[\left|\frac{2\alpha_{11}}{\sin(\theta)}\right|^2\frac{\cos(\theta)+1}{\sin(\theta)}-\cot\left(\frac{\overline{\theta}}{2}\right)\right]}\\ e^{i\overline{\phi}}\left[\left|\frac{2\alpha_{11}}{\sin(\theta)}\right|^2\frac{\cos(\theta)+1}{\sin(\theta)}-\cot\left(\frac{\overline{\theta}}{2}\right)\right] & e^{2\Im(\phi)}\left[\left|\frac{2\alpha_{11}}{\sin(\theta)}\right|^2+\left|\frac{\sin(\theta)}{\cos(\theta)-1}\right|^2\right]\end{pmatrix}.$$

Moreover, if we fix the parameters $\theta = \frac{\pi}{3} + \frac{i}{2}$, $\phi = \frac{\pi}{4} - i$, $E = 1$ in H_{MO} and $\alpha_{11} = 1$, we obtain the following representation of $S_\varphi^{1/2(-)}$ and $S_\Psi^{1/2(-)}$:

$$S_\varphi^{1/2(-)} = |N_\varphi| \begin{pmatrix} 1.076 & -0.709 - 0.005i \\ -0.709 + 0.005i & 4.532 \end{pmatrix},$$

$$S_\Psi^{1/2(-)} = \frac{1}{|N_\varphi|} \begin{pmatrix} 1.035 & 0.162 + 0.001i \\ 0.162 - 0.001i & 0.245 \end{pmatrix}.$$

The *self-adjoint counterpart* of the Hamiltonian H_{M0} is

$$h_{MO} = S_\Psi^{1/2(-)} H_{MO}\, S_\varphi^{1/2(-)} = \begin{pmatrix} 0.695 & 0.523 - 0.492i \\ 0.523 + 0.492i & -0.695 \end{pmatrix}.$$

D. A *relativistic* example

We now briefly consider the Hamiltonian introduced in [10] and further considered in [11]:

$$H_{rel} = \begin{pmatrix} mc^2 & cp_x + v \\ cp_x - v & -mc^2 \end{pmatrix}. \quad (3.16)$$

Here we are assuming that m, v, c, and p_x are all real quantities. If $c^2 p_x^2 \neq v^2$, then H_{rel} can be seen as a particular case of the Hamiltonian H_{MO}, fixing first $\theta = \arctan(\frac{c^2 p_x^2 - v^2}{m^2 c^4})$, then taking $E = \frac{mc^2}{\cos(\theta)}$ and finally $\phi = \arccos[\frac{cp_x}{E\sin(\theta)}]$. Something interesting happens if $c^2 p_x^2 = v^2 \neq 0$. In this case, it is easy to check that H_{rel} and H_{MO} are different for any possible choice of the parameters. This is because, while only one nondiagonal matrix element in H_{rel} can be different from zero, the analogous elements in H_{MO} are both zero or both not zero. Hence the two models, in this case, are really different. However, it is still possible that H_{rel} coincides with H in (2.12). And, in fact, we find that this is so if $cp_x = v$. In this case, it is enough to fix $\omega\gamma = 2v$, $\alpha = 0$, $\beta = \frac{mc^2}{v}$, and $\rho = mc^2$ or $\omega\gamma = 2v$, $\alpha = \frac{mc^2}{v}$, $\beta = 0$, and $\rho = -mc^2$. On the other hand, because of the asymmetry between the (1,2) and the (2,1) elements in H, there is no such possibility if $cp_x = -v$: in this case, PFs are absent.

If $cp_x \neq -v$, the eigenvectors of H_{rel}, its expression in terms of pseudofermionic operators, and the intertwining operators can all be deduced by adapting our general results to the present situation.

IV. CONCLUSIONS

We have shown how the general setting of PFs can be used in the analysis of different physical systems introduced over the years in connection with pseudo-Hermitian quantum mechanics. The procedure proposed here, other than being rather general and, in our opinion, useful for many other models, provides a set of simple rules and useful results linked to the anticommutation rules in (2.1). It could be worth mentioning that our analysis does not include all of the two-by-two matrices introduced over the years in our context. For instance, in [12], other examples are given, even in higher dimensions. However, the Hamiltonian $H_{JSM} = \begin{pmatrix} a & ib \\ ib & -a \end{pmatrix}$, mentioned in [12], where $a \in \mathbb{R}$ and $b \in \mathbb{R} \setminus \{0\}$, is a particular case of H_{MO}: we just have to take $\phi = \frac{\pi}{2}$, and then relate E and θ to a and b.

In our opinion, it is also interesting to stress that the existence of pseudofermionic operators appears to be deeply related to the existence of EPs: in fact, in all of the models considered here, a lack of validity of (2.1) implies coalescence of eigenvalues. This is expected, since a pseudofermionic structure is intrinsically connected with the existence of noncoincident eigenvalues. We believe this nice, simple result can be extended to more pseudofermionic modes (i.e., to Hilbert spaces with dimension 2^N, for some natural N) and to the much more complicated situation of pseudobosons, where (2.1) are replaced by a deformed version of canonical commutation rules [13]. This will be part of our future analysis.

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